

# COMMUTING DIFFERENTIAL AND DIFFERENCE OPERATORS ASSOCIATED TO COMPLEX CURVES, II

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**Introduction.** This paper is a sequel to [4]. Our main aim is to construct a commuting family of difference-evaluation operators  $(T_z^{(\Pi)})_z$ , deforming the difference-evaluation operators  $T_z^{class}$  of [4], and to interpret them as the action of the center of a quantum algebra in the space of intertwiners of a “regular” subalgebra.

Let us recall first some points of [4]. In that paper, we proposed a functional approach to the Knizhnik-Zamolodchikov-Bernard (KZB) connection, relying on the functional picture for conformal blocks of [9]. Recall that conformal blocks are associated to a complex curve  $X$  with a marked point  $P_0$ , a simple Lie algebra  $\bar{\mathfrak{g}}$  and representations  $\mathbb{V}$  and  $V$  of  $\mathfrak{g}$  and  $\mathfrak{g}^{out}$ , where  $\mathfrak{g}$  is the Kac-Moody algebra  $(\bar{\mathfrak{g}} \otimes \mathcal{K}) \oplus \mathbb{C}K$ , and  $\mathfrak{g}^{out}$  is the Lie subalgebra of  $\mathfrak{g}$  formed of the currents regular outside  $P_0$  (we denote by  $\mathcal{K}$  is the local field of  $X$  at  $P_0$ ), and defined as the space of  $\mathfrak{g}^{out}$ -intertwiners  $\psi$  from  $\mathbb{V}$  to  $V$ . Twisted conformal blocks are defined in the same way, replacing  $\mathfrak{g}^{out}$  by the Lie subalgebra  $\mathfrak{g}_{\lambda_0}^{out}$  of  $\mathfrak{g}$  formed of the maps  $x$  from the universal cover of  $X$ , regular outside the preimage of  $P_0$ , with transformation properties  $x(\gamma_{A_a} z) = x(z)$  and  $x(\gamma_{B_a} z) = e^{\lambda_a^{(0)}} x(z) e^{-\lambda_a^{(0)}}$ , where  $\gamma_{A_a}$  and  $\gamma_{B_a}$  are deck transformations corresponding to  $a$ - and  $b$ -cycles;  $\lambda_0 = (\lambda_a^{(0)})$  belongs to  $\bar{\mathfrak{h}}^g$ , where  $\bar{\mathfrak{h}}$  is the Cartan subalgebra of  $\bar{\mathfrak{g}}$  and  $\mathfrak{g}_{\lambda_0}^{out}$  is the Lie subalgebra of  $\mathfrak{g}$ . We parametrize the space of twisted conformal blocks by associating to  $\psi_{\lambda_0}$  the twisted correlation functions of currents of  $\mathfrak{g}$  associated to the simple root generators of its nilpotent subalgebra  $\bar{\mathfrak{n}}_+$ . Denote by  $e_i, f_i, h_i$  the Chevalley generators of  $\bar{\mathfrak{g}}$ , so that the  $e_i$  generate  $\bar{\mathfrak{n}}_+$ , and set  $x[f] = x \otimes f$ , for  $x$  in  $\bar{\mathfrak{g}}$ ,  $f$  in  $\mathcal{K}$ . To a vector  $v$  of  $\mathbb{V}$ , annihilated by the  $h_i[z^{k'}], f_i[z^{1-g+k}]$ ,  $k' > 0, k \geq 0$  (this property is shared by the extremal vectors in integrable modules), and to an intertwiner  $\psi_{\lambda_0}$ , weon  $\mathbb{V}$ ,  $\mathfrak{g}_{\lambda_0}^{out}$ -invariant, we associate the generating series

$$f(\lambda_a^{(i)} | z_j^{(i)}) = \langle \psi_{\lambda} [\prod_i \prod_{j=1}^{n_j} e_i(z_j^{(i)}) v], \xi \rangle,$$

where  $\psi_{\lambda} = \psi_{\lambda_0} \circ e^{\sum_a (\lambda_a - \lambda_a^{(0)}) h[r_a]}$ , the  $r_a$  are multivalued functions on  $X$ , constant along  $a$ -cycles and with additive constants along  $b$ -cycles, and  $\xi$  is a lowest weight form on  $V$ . The  $\lambda_a$  are formal parameters near  $(\lambda_a^{(0)})$  and the  $z_i$  are formal parameters near  $P_0$ .

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In [4], we expressed the KZB connection in terms of these correlation functions. Let  $T(z)$  denote the Sugawara tensor; it is a series in  $(U\mathfrak{g})_{loc}[[z, z^{-1}]]$ , where  $(U\mathfrak{g})_{loc}$  is the local completion of the universal enveloping algebra  $U\mathfrak{g}$ . In the case  $\bar{\mathfrak{g}} = \mathfrak{sl}_2$ , we have

**Theorem 0.1.** (see [4]) *Let  $T_z^{class}$  be the differential-evaluation operator acting on functions  $f(\lambda_1, \dots, \lambda_g | z_1, \dots, z_n)$  as*

$$\begin{aligned} T_z^{class} = & \frac{1}{2} \left[ \sum_a \omega_a(z) \partial_{\lambda_a} + 2 \sum_i G^{(I)}(z, z_i) - \sum_j \Lambda_j G^{(I)}(z, P_j) \right]^2 \\ & + \sum_a D_z^{(2\lambda)} \omega_a(z) \partial_{\lambda_a} + 2 \sum_i D_z^{(2\lambda)} G^{(I)}(z, z_i) - \sum_j \Lambda_j D_z^{(2\lambda)} G^{(I)}(z, P_j) + k\omega_{2\lambda}(z) \\ & + \sum_{i=1}^n \left( -2G_{2\lambda}^{(I)}(z, z_i) \left[ \sum_a \omega_a(z_i) \partial_{\lambda_a} + 2 \sum_{j \neq i} G^{(I)}(z_i, z_j) - \sum_k \Lambda_k G^{(I)}(z_i, P_k) \right] \right. \\ & \left. - 4G_{2\lambda}^{(I)}(z, z_i) G^{(I)}(z_i, z) + 2kd_{z_i} G_{2\lambda}^{(I)}(z, z_i) \right) \circ \text{ev}_z^{(i)}, \end{aligned}$$

where

$$(\text{ev}_z^{(i)} f)(\lambda | z_1, \dots, z_n) = f(\lambda | z_1, \dots, z, \dots, z_n)$$

( $z$  in  $i$ th position), we set  $\lambda = (\lambda_1, \dots, \lambda_g)$ ,  $D_z^{(2\lambda)}$  is a connection on the bundle  $K$  of differentials on  $X$  and has simple pole at  $P_0$ , the  $\omega_a$  are the holomorphic one-forms associated with the  $a$ -cycles,  $\omega_{2\lambda}$  is a quadratic differential with double poles at  $P_0$ ,  $G^{(I)}$  and  $G_{\lambda}^{(I)}$  are (twisted) Green functions,  $P_j$  are some point of  $X - \{P_0\}$  and  $\Lambda_i$  are some numbers.

If  $V$  is the product  $\otimes_i V_{-\Lambda_i}(P_i)$  of evaluation modules ( $V_{-\Lambda_i}$  is the  $\mathfrak{sl}_2$ -module with lowest weight  $-\Lambda_i$ ), we have the equality

$$\langle \psi_{\lambda}[T(z) \prod_i e_i(z_j^{(i)})v], \xi \rangle = (T_z^{class} f)(\lambda | z_1, \dots, z_n),$$

if  $k$  is the level of  $\mathbb{V}$ . The operators  $T_z^{class}$  commute when  $k = -2$ .

When  $X$  is  $\mathbb{CP}^1$ , the expression of  $T_z^{class}$  is similar to the expression for the action of the Hamiltonians on Bethe vectors obtained in the Bethe ansatz approach to the Gaudin system (see [10]).

In the present paper, we repeat these steps of [4] in the quantum case, at the critical level. We replace the Kac-Moody algebra  $\mathfrak{g}$  by the quantum group  $U_{\hbar, \omega} \mathfrak{g}$  associated to a pair  $(X, \omega)$  of a curve  $X$  and a rational differential  $\omega$  ([6]). The relations for this algebra depend on the choice of a Lagrangian subspace of  $\mathcal{K}$ , that we construct in sect. 1. We recall the presentation of  $U_{\hbar, \omega} \mathfrak{g}$  in terms of generating fields  $e(z), f(z), k^{\pm}(z)$  (sect. 2). The algebra  $U_{\hbar, \omega} \mathfrak{g}$  contains a subalgebra  $U_{\hbar} \mathfrak{g}^{out}$ , which is a flat deformation of the enveloping algebra of  $\mathfrak{g}^{out}$  ([7]).

Let  $\bar{\mathfrak{g}} = \bar{\mathfrak{n}}_+ \oplus \bar{\mathfrak{h}} \oplus \bar{\mathfrak{n}}_-$  be the Cartan decomposition of  $\bar{\mathfrak{g}}$ . Let  $\mathfrak{m}$  be the maximal ideal at  $P_0$  and  $\mathfrak{b}_{in}$  be the subalgebra of  $\mathfrak{g}$  defined as  $\mathfrak{b}_{in} = (\bar{\mathfrak{h}} \otimes \mathfrak{m}) \oplus (\bar{\mathfrak{n}}_+ \otimes \mathcal{K})$ .

We construct, in  $U_{\hbar,\omega}\mathfrak{g}$ , a subalgebra isomorphic to  $(U\mathfrak{b}_{in})[[\hbar]]$  (sect. 3). This subalgebra is expressed in terms of “new” generating fields  $\tilde{e}(z)$ ,  $\tilde{f}(z)$  and  $k_{tot}^\pm(z)$ ; we study their relations in sect. 4. In the rational case, such generating fields appeared in [15]. We express a generating function for central elements  $T(z)$  deforming the Sugawara tensor by the formula (see Thm. 5.1)

$$T(z) =: e(z)\tilde{f}(z) :_\lambda + a_\lambda(z)k_{tot}^+(z) + b_\lambda(z)k_{tot}^-(z),$$

where  $:a(z)b(z):_\lambda$  denotes a normal ordered product, depending on  $\lambda$  and  $a_\lambda(z)$  and  $b_\lambda(z)$  are formal series of  $\mathcal{K}[[\lambda_a - \lambda_a^{(0)}]][[\hbar]]$  defined by (48) and (49). We also obtain another expression for  $T(z)$  of the type obtained in [12, 14], see (6).

We construct a subalgebra  $U_{\hbar}\mathfrak{g}_{\lambda_0}^{out}$  of  $U_{\hbar,\omega}\mathfrak{g}$  in sect. 6 deforming the enveloping algebra of  $\mathfrak{g}_{\lambda_0}^{out}$  and study a class of its representations (sect. 7). We show that such representations have a lowest weight form  $\xi$ , such that

$$\xi \circ f[r_{-2\lambda_0}] = 0, \quad \xi \circ k^+(z) = \pi(z)\xi,$$

for  $r_{-2\lambda_0}$  in  $R_{-2\lambda_0}$  and  $\pi(z)$  a formal series, which is an analogue of the Drinfeld polynomial.

To a module  $\mathbb{V}$  over  $U_{\hbar,\omega}\mathfrak{g}$ , and to a morphism  $\psi_{\lambda_0} : \mathbb{V} \rightarrow V$  of  $U_{\hbar}\mathfrak{g}_{\lambda_0}^{out}$ -modules, where  $V$  is a product of evaluation modules, we associate the correlation function

$$f(\lambda_1, \dots, \lambda_g | z_1, \dots, z_n) = \langle \psi_\lambda[\tilde{e}(u_1) \cdots \tilde{e}(u_n)v], \xi \rangle,$$

where  $\xi$  is a lowest weight form on  $V$  and  $\psi_\lambda = \psi_{\lambda_0} \circ e^{\sum_i (\lambda_a - \lambda_a^{(0)})h[r_a]}$ . We study the functional properties of  $f(\lambda_1, \dots, \lambda_g | z_1, \dots, z_g)$  in sect. 8.

Our main result is then

**Theorem 0.2.** (see Thm. 10.1) Let for any formal series  $\Pi$ ,  $(T_z^{(\Pi)})_z$  be the family of operators acting on functions  $f(\lambda | u_1, \dots, u_n)$ , defined as

$$\begin{aligned} T_z^{(\Pi)} &= \Pi(z)a'_\lambda(z | u_1, \dots, u_n) \circ e^{\sum_a \omega'_a(z)\partial/\partial\lambda_a} + \Pi(q^{-\partial}z)^{-1}a''_\lambda(z | u_1, \dots, u_n) \circ e^{\sum_a \omega''_a(z)\partial/\partial\lambda_a} \\ &+ \sum_i \Pi(u_i)c_\lambda^{(i)}(z | u_1, \dots, u_n) \circ e^{\sum_a \omega'_a(z)\partial/\partial\lambda_a} \circ \text{ev}_z^{(i)} \\ &+ \sum_i \Pi(q^{-\partial}u_i)^{-1}c_\lambda^{(i)}(z | u_1, \dots, u_n) \circ e^{\sum_a \omega''_a(z)\partial/\partial\lambda_a} \circ \text{ev}_z^{(i)}, \end{aligned}$$

where the multiplication operators are denoted as functions, and we set

$$a'_\lambda(z | u_1, \dots, u_n) = a_\lambda(z) \prod_i q_m(z, u_i), \quad a''_\lambda(z | u_1, \dots, u_n) = b'_\lambda(z)\kappa(z) \prod_i q_m(q^{-\partial}z, u_i)^{-1},$$

$$c_\lambda^{(i)}(z | u_1, \dots, u_n) = -\frac{1}{\hbar}G_{2\lambda}(z, u_i)q_m(u_i, z) \prod_{j \neq i} q_m(u_i, u_j),$$

$$c_\lambda^{(i)}(z | u_1, \dots, u_n) = \frac{1}{\hbar}G_{2\lambda}(z, q^{-\partial}u_i)\kappa(u_i)q_m(q^{-\partial}u_i, z)^{-1} \prod_{j \neq i} q_m(q^{-\partial}u_i, u_j)^{-1},$$

$$\omega'_a = \hbar \frac{1}{1+q^{-\partial}}(\omega_a/\omega)(z), \quad \omega''_a = -\hbar \frac{1}{1+q^{\partial}}(\omega_a/\omega)(z), \quad G_{2\lambda}(z, w) = G_{2\lambda}^{(I)}(z, w)/\omega(z),$$

where  $\partial$  is the derivation associated with  $\omega$ , so that  $\partial f = df/\omega(z)$ ,  $a_\lambda$ ,  $b'_\lambda$ ,  $q_m$  and  $\kappa$  are defined in (48), (50), (59) and (60), and  $q = e^\hbar$ . The operators  $T_z^{(\Pi)}$  commute and normalize first order difference operators  $\hat{f}[\rho]$  defined by (63). Moreover, we have, if the subalgebra  $U_\hbar \mathfrak{b}^{\geq 1-g}$  of  $U_\hbar \mathfrak{b}_{in}$  acts on  $v$  by the character  $\chi_n$  (see sect. 10),

$$\langle \psi_\lambda [T(z) \tilde{e}(u_1) \cdots \tilde{e}(u_n) v], \xi \rangle = T_z \{ \langle \psi_\lambda [\tilde{e}(u_1) \cdots \tilde{e}(u_n) v], \xi \rangle \},$$

where  $\Pi$  can be expressed in terms of  $\pi$ . We also set  $\Pi(q^\partial z) = (q^\partial \Pi)(z)$ .

The  $T_z^{(\Pi)}$  are difference deformations of the  $T_z^{class}$ . In the rational case, we identify the operators  $T_z^{(\Pi)}$  with the commuting family of operators provided by the Yangian action on the hypergeometric spaces of [16] (see sect. 11). In the elliptic case, we identify  $T_z^{(\Pi=1)}$  with the first  $q$ -Lamé operator (rem. 11).

Let us say some words about possible prolongations of the present work:

1) noncritical level. One could try to prove analogues of the theta-behavior results of [4] for the twisted correlation functions of integrable modules over  $U_{\hbar, \omega} \mathfrak{g}$ . Another problem is to find analogues of the KZB flows for noncritical level, by extending the approach of [13] to the quasi-Hopf situation.

2) versions where  $\hbar$  takes complex values. When  $\omega$  is the pull-back of the form  $dz$  or  $dz/z$  from a morphism  $X \rightarrow \mathbb{CP}^1$  or  $X \rightarrow E$ ,  $E$  some elliptic curve,  $q^\partial$  at least makes sense as some correspondence on  $X$ . It could then be possible to find a presentation of  $U_{\hbar, \omega} \mathfrak{g}$  allowing for complex values of  $\hbar$ . This was done in [8] in the case  $X = \mathbb{CP}^1$ ,  $\omega = z^N dz$ .

3) Bethe ansatz for the operators  $T_z$ . In [10], Bethe equations for the Gaudin system are shown to be equivalent to the existence of intertwining operators at critical level, and in turn to a trivial monodromy condition for some connection. The similar study should be possible for the systems constructed here, so that they could be viewed as  $q$ -deformations of the differential systems arising in [1].

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## 1. GEOMETRIC SETTING

**1.1. Isotropic supplementaries.** Let  $X$  be a smooth compact complex curve of genus  $g$ , endowed with a nonzero holomorphic form  $\omega$ . Let  $\sum_{i=1}^p n_i P_i$  be the divisor of  $\omega$  (we have  $n_i > 0$ ,  $\sum_i n_i = 2(g-1)$ ). Let for each  $i$ ,  $\mathcal{K}_i$  be the local field at  $P_i$ ,  $\mathcal{O}_i$  the local ring at this point and  $\mathfrak{m}_i$  the maximal ideal of  $\mathcal{O}_i$ . Define  $\mathcal{K}$  as  $\oplus_{i=1}^p \mathcal{K}_i$  and  $R$  as the space of rational functions on  $X$ , regular outside  $\{P_i\}$ ; we view it as a subring of  $\mathcal{K}$ . For each  $i$ , let  $z_i$  be a local coordinate at  $P_i$ . Then  $\mathcal{O}_i = \mathbb{C}[[z_i]]$ ,  $\mathfrak{m}_i = z_i \mathcal{O}_i$  and  $\mathcal{K}_i = \mathbb{C}((z_i))$ .

$\mathcal{K}$  is endowed with a scalar product  $\langle \cdot, \cdot \rangle_{\mathcal{K}}$  defined by

$$\langle f, g \rangle_{\mathcal{K}} = \sum_{i=1}^p \text{res}_{P_i}(fg\omega).$$

Let us fix on  $X$  a choice of  $a$ - and  $b$ -cycles  $(A_a)_{1 \leq a \leq g}$  and  $(B_a)_{1 \leq a \leq g}$ . Let  $\tilde{X}$  be the universal cover of  $X$  and  $\pi : \tilde{X} \rightarrow X$  be the cover map. Denote by  $\gamma_{A_a}$  and  $\gamma_{B_a}$  the deck transformations associated with the cycles  $A_a$  and  $B_a$ .

**Lemma 1.1.** *There exists a linearly independent family of  $R$  formed by elements  $f_{(m_i)}$ , where  $m_i, i = 1, \dots, p$  are integers such that  $m_i \geq n_i$  for each  $i$  and  $\sum_i m_i \geq \sum_i n_i + 2$ , with  $\text{val}_{P_i}(f_{(m_i)}) = -m_i$ , for each  $i = 1, \dots, p$ .*

*Proof.* Let us first construct the  $f_{(m_i)}$ . Assume  $m_j \geq n_j + 1$ , then by the Riemann-Roch theorem,

$$h^0(\mathcal{O}(\sum_i m_i P_i)) - h^0(\mathcal{O}(\sum_i m_i P_i - P_j)) = 1 + h^1(\mathcal{O}(\sum_i m_i P_i)) - h^1(\mathcal{O}(\sum_i m_i P_i - P_j));$$

by Serre duality this is equal to

$$1 + h^0(\mathcal{O}(\sum_i (n_i - m_i) P_i)) - h^0(\mathcal{O}(\sum_i (n_i - m_i) P_i + P_j)).$$

All the  $n_i - m_i + \delta_{ij}$  are  $\leq 0$  and their sum is  $< 0$ , so not all of them are zero. Therefore both  $h^0$  vanish. This proves the existence of the  $f_{(m_i)}$ .  $\square$

**Lemma 1.2.** *We have  $g$  functions  $r_a$  defined on  $\tilde{X}$ , regular outside  $\pi^{-1}(\{P_i\})$ , such that*

- i)  $r_a \circ \gamma_{B_b} = r_a - \delta_{ab}$ ,
- ii)  $\text{val}_{P_i}(r_a) \geq -n_i - \delta_{i1}$  and
- iii)  $\int_{A_a} r_b \omega = \frac{1}{2} \int_{A_a} \omega \delta_{ab}$ .

*Proof.* The existence of rational functions  $\tilde{r}_a$  defined on  $\tilde{X}$ , regular outside  $\pi^{-1}(\{P_i\})$  and satisfying i) is a consequence of [4] Cor. 1.1. Adding to them suitable combinations of the  $f_{(m_i)}$ , one gets functions  $\bar{r}_a$  satisfying both i) and ii). Let  $\omega_a$  be a basis of the space of holomorphic one-forms on  $X$ . The ratios  $\omega_a/\omega$  are elements of  $R$ , with valuation at each  $P_i$  less or equal to  $-n_i$ . Adding to the  $\bar{r}_a$  suitable combinations of the  $\omega_a/\omega$ , one obtains elements  $r_a$  satisfying i), ii) and iii).  $\square$

**Proposition 1.1.** *Set  $\Lambda = (\oplus_a \mathbb{C} r_a) \oplus (\mathfrak{m}_1 \oplus \mathcal{O}_2 \oplus \dots \oplus \mathcal{O}_p)$ . We have a direct sum decomposition*

$$\mathcal{K} = R \oplus \Lambda;$$

*moreover,  $R$  and  $\Lambda$  are both maximal isotropic subspaces of  $\mathcal{K}$ .*

*Proof.* The fact that  $\mathcal{K} = R \oplus \Lambda$  follows from [4], Prop. 1.1. That  $\langle r_a, r_b \rangle = 0$  follows from [8], 4.1.1. Let us show that  $\langle r_a, \mathbf{m}_1 \rangle$  vanishes: for  $n > 0$ ,  $\text{val}_{P_1}(r_a z_1^n \omega) > (-n_1 - 1) + n_1 = -1$  so  $\text{res}_{P_1}(r_a z_1^n \omega)$  is zero; and for  $i > 1$ ,  $\langle r_a, \mathcal{O}_i \rangle$  vanishes because for  $n \geq 0$ ,  $\text{val}_{P_i}(r_a z_i^n \omega) \geq -n_i + n_i = 0$  so  $\text{res}_{P_i}(r_a z_i^n \omega)$  is zero.  $\square$

*Remark 1.* In the case where  $\omega$  has a unique zero of order  $2(g-1)$  at some point  $P_0$ ,  $R$  is spanned by  $f_0, f_{-a_1}, f_{-a_2}, \dots, f_{-a_{g-1}}, f_{-2g}, f_{-2g-1}, f_{-2g-2}, \dots$  with  $\text{val}_{P_0}(f_i) = i$ : if  $\omega_1, \dots, \omega_g$  be a basis of the space of holomorphic one-forms  $H^0(X, \Omega_X)$ , with  $\text{val}_{P_0}(\omega_i) = b_i$ , so that  $0 \leq b_1 < b_2 < \dots < b_g = 2(g-1)$ , then  $f_0 = 1$ ,  $f_{-a_1} = \omega_{g-1}/\omega_g$ ,  $f_{-a_2} = \omega_{g-1}/\omega_g$ , etc. On the other hand, the  $r_a$  may be chosen to have poles of order  $b_1, \dots, b_g$  at  $P_0$ , with  $\{a_1, \dots, a_{g-1}\} \cup \{b_1, \dots, b_g\} = \{1, \dots, 2g-1\}$ .

If  $X$  is a hyperelliptic curve  $y^2 = P_{2g+1}(x)$ ,  $P_{2g+1}$  a polynomial of degree  $2g+1$ , and  $\omega = dx/y$ ; more generally, if  $X$  is a plane curve of equation  $P(z) = Q(y)$ , and  $\omega = dx/Q'(y) = -dy/P'(x)$ , with  $P$  and  $Q$  generic polynomials of coprime degrees  $p$  and  $q$ , (in that case,  $g = \frac{(p-1)(q-1)}{2}$ ),  $\omega$  has a zero of order  $2(g-1)$  at the point at infinity.

**1.2. (Twisted) Green functions.** We will denote by  $z$  the  $n$ -uple  $(z_i)$  of  $\mathcal{K}$ . We will denote by  $\mathbb{C}[[z, z^{-1}]]$  the set of series  $\sum_{i=1}^p \sum_{n \in \mathbb{Z}} a_{in} z_i^n$ , and by  $\mathbb{C}[[z, w]]$  the space  $\prod_{1 \leq i, j \leq p} \mathbb{C}[[z_i, w_j]]$ .

We define  $\delta(z, w)$  as the sum  $\sum_i \epsilon^i(z) \epsilon_i(w)$ , where  $(\epsilon^i)$  and  $(\epsilon_i)$  are dual bases of  $\mathcal{K}$  for  $\langle, \rangle_{\mathcal{K}}$ .

The space of functions in two variables  $z$  and  $w$  will be identified with the tensor square of the space of functions in one variable, via the identification  $a(z)b(w) \mapsto a \otimes b$ .

**1.2.1. Green function.** Let  $(e^i), (e_i)$  be dual bases of  $R$  and  $\Lambda$ . We will assume that  $(e_i)$  is the union of  $(r_a)$  and a basis of  $\mathbf{m} = \mathbf{m}_1 \oplus \mathcal{O}_2 \oplus \dots \oplus \mathcal{O}_p$ . We set

$$G = \sum_i e^i \otimes e_i. \quad (1)$$

We have then  $\delta(z, w) = G(z, w) + G(w, z)$ .  $G(z, w)$  is the collection of expansions, for  $w$  near each  $P_i$ , of a rational function defined on  $X^2$ , antisymmetric in  $z$  and  $w$ , regular except for poles when  $z$  or  $w$  meets some  $P_i$  and a simple pole at the diagonal.

**1.2.2. Twisted Green functions.** To  $\lambda_0 = (\lambda_a^{(0)})_{1 \leq a \leq g}$  a vector of  $\mathbb{C}^g$  is associated the line bundle  $\mathcal{L}_{2\lambda_0}$  over  $X$ . The space  $H^0(X - \{P_0\}, \mathcal{L}_{2\lambda_0})$  may be identified with the space of functions on  $\tilde{X}$ , regular outside  $\pi^{-1}(P_0)$ , with transformation properties

$$f(\gamma_{B_a} z) = f(z) \quad \text{and} \quad f(\gamma_{B_a} z) = e^{-2\lambda_a^{(0)}} f(z). \quad (2)$$

This space of functions will be denoted  $R_{-2\lambda_0}$ . For  $\lambda_0$  generic, a complement in  $\mathcal{K}$  of this space is  $z_1^{1-g} \mathcal{O}_1 \oplus \mathcal{O}_2 \oplus \dots \oplus \mathcal{O}_p$ .

Let  $\lambda = (\lambda_a)_{1 \leq a \leq g}$  be  $g$  formal parameters at the vicinity of  $\lambda_0$ , and define  $R_{-2\lambda}$  as the  $\mathbb{C}[[\lambda_a - \lambda_a^{(0)}]]$ -submodule of  $\mathcal{K}[[\lambda_a - \lambda_a^{(0)}]]$  generated by the  $e^{2\sum_a (\lambda_a - \lambda_a^{(0)})r_a} \phi$ ,  $\phi \in R_{-2\lambda_0}$ . Define also  $\Lambda' = (z_1^{1-g} \mathcal{O}_1 \oplus \mathcal{O}_2 \oplus \cdots \oplus \mathcal{O}_p)[[\lambda_a - \lambda_a^{(0)}]]$ .

Then we have a direct sum decomposition

$$\mathcal{K}[[\lambda_a - \lambda_a^{(0)}]] = R_{-2\lambda} \oplus \Lambda'.$$

For  $\phi$  in  $\mathcal{K}[[\lambda_a - \lambda_a^{(0)}]]$ , we denote by  $\phi_{\Lambda'}$  and  $\phi_{R_{2\lambda}}$  the projections of  $\phi$  on  $\Lambda'$  parallel to  $R_{2\lambda}$ , resp. on  $R_{2\lambda}$  parallel to  $\Lambda'$ . For  $\phi(z)$  a series  $\sum_i \phi_i \epsilon_i(z)$ , we define  $\phi(z)_{z \rightarrow \Lambda'}$  as  $\sum_i \phi_i(\epsilon_i)_{\Lambda'}(z)$  and  $\phi(z)_{z \rightarrow R_{2\lambda}}$  as  $\sum_i \phi_i(\epsilon_i)_{R_{2\lambda}}(z)$ .

We have then  $f(z) = f(z)_{z \rightarrow \Lambda'} + f(z)_{z \rightarrow R_{2\lambda}}$ .

Let  $(e_{2\lambda}^i)$ ,  $(e'_i)$  be dual bases of  $R_{2\lambda}$  and  $\Lambda'$ , and let us set

$$G_{2\lambda}(z, w) = \sum_i e_{2\lambda}^i(z) e'_i(w). \quad (3)$$

We have  $\delta(z, w) = G_{2\lambda}(z, w) + G_{-2\lambda}(w, z)$ .  $G_{-2\lambda}(z, w) - G(z, w)$  belongs to  $\mathbb{C}[[z, w]][z^{-1}, w^{-1}]$ . The functions

$$g_{\lambda}^+(z) = (G_{-2\lambda} - G)(q^{\partial} z, z), \quad g_{\lambda}^-(z) = (G_{-2\lambda} - G)(q^{-\partial} z, z)$$

then belong to  $\mathcal{K}[[\hbar]]$ .

*Remark 2. Relation with the Green functions of [4].* In [4], we introduced Green function  $G(z, w)$  and a twisted Green function  $G_{2\lambda}(z, w)$ , that we denote here by  $G^{(\text{I})}(z, w)$  and  $G_{2\lambda}^{(\text{I})}(z, w)$ . Let  $\omega_a$  be the basis of one-forms on  $X$ , associated to  $(A_a)_{1 \leq a \leq g}$ . The relation of these Green functions with  $G(z, w)$  and  $G_{2\lambda}(z, w)$  defined by (1) (under the assumptions of Prop. 1.1) and (3) is

$$G(z, w) = \left( G^{(\text{I})}(z, w) + \sum_a \omega_a(z) r_a(w) \right) / \omega(z)$$

and

$$G_{2\lambda}(z, w) = G_{2\lambda}^{(\text{I})}(z, w) / \omega(z).$$

Set

$$\bar{g}_{\lambda}(z) = \lim_{z \rightarrow w} (G_{\lambda}(z, w) - G(z, w)).$$

One can show that

$$\bar{g}_{\lambda}(z) = \sum_a \partial_{\epsilon_a} \ln \Theta(-\lambda + (g-1)P_0 - \Delta) \omega_a(z) / \omega(z).$$

where  $(\epsilon_a)_{1 \leq a \leq g}$  is the canonical basis of  $\mathbb{C}^g$ ,  $\Theta$  is the Riemann theta-function on the Jacobian of  $X$  and points of  $X$  are identified with their images by the Abel-Jacobi map ([4], 4.4).

2. THE ALGEBRA  $U_{\hbar, \omega} \mathfrak{g}$ 

**Notation.** For  $E$  a vector space and  $E', E''$  two subspaces, such that  $E$  is the direct sum  $E' \oplus E''$ , and for  $\phi$  in  $E$ , we denote by  $\phi_{E' \parallel E''}$  the projection of  $\phi$  on  $E'$  parallel to  $E''$ . In the case of the decompositions  $\mathcal{K} = R \oplus \Lambda$ ,  $\mathcal{K}[[\lambda_a - \lambda_a^{(0)}]] = R_{2\lambda} \oplus \Lambda'$  and  $\mathcal{K} = R_{(a)} \oplus \mathfrak{m}$  below, we will simply denote  $\phi_{E' \parallel E''}$  and  $\phi_{E'' \parallel E'}$  by  $\phi_{E'}$  and  $\phi_{E''}$  respectively.

For  $(\epsilon_i)$  a basis of  $\mathcal{K}$  and  $f(z)$  a series  $\sum_i \epsilon_i(z) \otimes v_i$  in some completion of  $\mathcal{K} \otimes V$ ,  $V$  some vector space, we define  $f(z)_{z \rightarrow \Lambda}$  as  $\sum_i (\epsilon_i)_\Lambda(z) \otimes v_i$ , and  $f(z)_{z \rightarrow R}$  as  $\sum_i (\epsilon_i)_R(z) \otimes v_i$ . One define in the same way  $f(z)_{z \rightarrow R_{2\lambda}}$  and  $f(z)_{z \rightarrow \Lambda'}$ . If  $f(z, w)$  is a series  $\sum_{i,j} \epsilon_i(z) \epsilon_j(w) v_{ij}$ ,  $f(z, w)_{z \rightarrow \Lambda}$  is  $\sum_{i,j} (\epsilon_i)_\Lambda(z) \epsilon_j(w) v_{ij}$ , etc.

If  $f(z, w)$  belongs to  $R_z((w))$  (the space of series  $\sum_{i \geq n_0} r_i(z) w^i$ , with  $r_i$  in  $R$ ) and there exists  $g(z, w)$  in  $R_w((z))$ , such that  $f + g = (\pi \otimes id) \delta(z, w)$ , where  $\pi$  is some differential operator, then  $f$  is the expansion of a rational function on  $X - \{P_i\}^2$  with only poles at the diagonal, and  $g$  may be viewed as the analytic prolongation of  $-f$  in the region  $z \ll w$ . We write  $g(z, w) = -f(z, w)_{z \ll w}$ .

We write  $\partial_z$  for  $\partial \otimes id$ ,  $\partial_w$  for  $id \otimes \partial$ . we set  $\phi^{(21)}(z, w) = \phi(w, z)$ .

## 2.1. Results on kernels. (see [6])

We have

$$\partial_z G(z, w) = -G(z, w)^2 - \gamma,$$

for some  $\gamma \in R \otimes R$ .

Let  $\phi, \psi$  belong to  $\hbar \mathbb{C}[\gamma_0, \gamma_1, \dots][[\hbar]]$  such that

$$\partial_\hbar \psi = D\psi - 1 - \gamma_0 \psi^2, \quad \partial_\hbar \phi = D\phi - \gamma_0 \psi.$$

Here  $D = \sum_{i \geq 0} \gamma_{i+1} \partial_{\gamma_i}$ . we have

$$\psi(\hbar, \partial_z^i \gamma) = -\hbar + o(\hbar), \quad \phi(\hbar, \partial_z^i \gamma) = \frac{1}{2} \hbar^2 \gamma_0 + o(\hbar^2).$$

Set  $G^{(21)}(z, w) = G(w, z)$ . From identity (3.11) of [6] (with  $\partial$  transformed to  $-\partial$ ) and by (3.8) of [6], we have

$$\sum_i \frac{1 - q^{-\partial}}{\partial} e_i(z) \otimes e^i(w) = -\phi(-\hbar, \partial_z^i \gamma) + \ln(1 + G^{(21)} \psi(-\hbar, \partial_z^i \gamma)). \quad (4)$$

Set  $T = \frac{\sinh \hbar \partial}{\hbar \partial}$ . Let  $\tau$  in  $(R \otimes R)[[\hbar]]$  satisfy

$$\tau + \tau^{(21)} = - \sum_i e^i \otimes (T e_i)_R. \quad (5)$$

Let  $U$  be the linear map from  $\Lambda$  to  $R[[\hbar]]$  such that  $\tau = \sum_i U e_i \otimes e^i$ . We have

$$\sum_i (T + U) e_i \otimes e^i + \sum_i e^i \otimes (T + U) e_i = (T \otimes id) \delta(z, w),$$

which means that after analytic prolongation the sum  $\sum_i (T + U) e_i \otimes e^i$  is anti-symmetric in  $z$  and  $w$ .

Set  $T_+ = \frac{1-q^{-\partial}}{2\hbar\partial}$  and define  $U_+ : \Lambda \rightarrow R[[\hbar]]$  by the formula  $U_+ = (1+q^\partial)^{-1} \circ U$ ; we have

$$(T_+ + U_+)(\lambda) = \frac{1}{1+q^\partial}((T+U)(\lambda)).$$

Define  $q_+$  by

$$q_+(z, w) = q^{2\sum_i (T_+ + U_+)e_i(z) \otimes e^i(w)},$$

it then follows from (4) that

$$q_+(z, w) = q^{2\sum_i (U_+ e_i)(z) \otimes e^i(w)} e^{-\phi(-\hbar, \partial_z^i \gamma)} (1 + G^{(21)} \psi(-\hbar, \partial_z^i \gamma)). \quad (6)$$

*Remark 3.* Formulas of this section correct a sign mistake in [6]: in sect. 3 of that paper,  $\partial$  should be changed to  $-\partial$ .

2.2. The algebra  $U_{\hbar, \omega} \mathfrak{g}$  has generators  $h^+[r], h^-[\lambda], e[\epsilon], f[\epsilon]$  and  $K$ , with  $r$  in  $R$ ,  $\lambda$  in  $\Lambda$  and  $\epsilon$  in  $\mathcal{K}$ ; generating series

$$x(z) = \sum_i x[\epsilon^i] \epsilon_i(z), \quad h^+(z) = \sum_i h^+[e^i] e_i(z), \quad h^-(z) = \sum_i h^-[e_i] e^i(z),$$

$x = e, f$ , and relations

$$x[\alpha\epsilon + \epsilon'] = \alpha x[\epsilon] + x[\epsilon'],$$

for  $\alpha$  scalar,  $x = h^+$ ,  $\epsilon, \epsilon'$  in  $R$ ;  $x = h^-$ ,  $\epsilon, \epsilon'$  in  $\Lambda$ ; or  $x = e, f$ ,  $\epsilon, \epsilon'$  in  $\mathcal{K}$ ;

$$[h^+[r], h^+[r']] = 0, \quad (7)$$

$$[K, \text{anything}] = 0, \quad [h^+[r], h^-[\lambda]] = \frac{2}{\hbar} \langle (1 - q^{-K\partial})r, \lambda \rangle, \quad (8)$$

$$[h^-[\lambda], h^-[\lambda']] = \frac{2}{\hbar} (\langle T((q^{K\partial}\lambda)_R), q^{K\partial}\lambda' \rangle + \langle U\lambda, \lambda' \rangle - \langle U((q^{K\partial}\lambda)_\Lambda), q^{K\partial}\lambda' \rangle) \quad (9)$$

$$[h^+[r], e(w)] = 2r(w)e(w), \quad [h^-[\lambda], e(w)] = 2[(T+U)(q^{K\partial}\lambda)_\Lambda](w)e(w), \quad (10)$$

$$[h^+[r], f(w)] = -2r(w)f(w), \quad [h^-[\lambda], f(w)] = -2[(T+U)\lambda](w)f(w), \quad (11)$$

$$(\alpha(z) - \alpha(q^{-\partial}w))e(z)e(w) = (\alpha(z) - \alpha(q^{-\partial}w))q^{2\sum_i (T+U)e_i(z) \otimes e^i(w)} e(w)e(z) \quad (12)$$

and

$$(\alpha(z) - \alpha(q^\partial w))f(z)f(w) = (\alpha(z) - \alpha(q^\partial w))q^{-2\sum_i (T+U)e_i(z) \otimes e^i(w)} f(w)f(z) \quad (13)$$

for any  $\alpha$  in  $\mathcal{K}$ ,

$$[e(z), f(w)] = \frac{1}{\hbar} [\delta(z, w) q^{((T+U)h^+)(z)} - (q^{-K\partial} \delta(z, w)) q^{-h^-(w)}], \quad (14)$$

$\phi(z, w) = \phi(\hbar, \partial_z^i \gamma)$ ,  $r, r'$  in  $R$ ,  $\lambda, \lambda'$  in  $\Lambda$  (see [6]).  $U_{\hbar, \omega} \mathfrak{g}$  is completed with respect to the topology defined by the left ideals generated by the  $x[\epsilon]$ ,  $\epsilon$  in  $\oplus_i z_i^N \mathcal{O}_i$ . The critical case corresponds to  $K = -2$ .

*Remark 4.* Relations (12) and (13) can be written

**2.3. Cartan currents.** In case we have a relation

$$a(z)b(w)a(z)^{-1} = \mu(z, w)b(w),$$

with  $a(z), b(w)$  currents of  $U_{\hbar, \omega} \mathfrak{g}$  and  $\mu(z, w)$  in  $\mathbb{C}((z))((w))[[\hbar]]$  or  $\mathbb{C}((w))((z))[[\hbar]]$ , we will define  $(a(z), b(w))$  as  $\mu(z, w)$ .

Set  $K^+(z) = q^{(T+U)h^+(z)}$ ,  $K^-(z) = q^{-h^-(z)}$ . Let us also set

$$q(z, w) = q^{2\sum_i (T+U)e_i(z) \otimes e^i(w)},$$

we have  $q(z, w) = (q(w, z)^{-1})_{w < z}$ . Then the relations involving Cartan generators can be expressed as

$$(K^+(z), K^+(w)) = 1, \quad (K^+(z), K^-(w)) = \frac{q(z, q^{-K\partial}(w))}{q(z, w)}, \quad (15)$$

$$(K^-(z), K^-(w)) = \frac{q(q^{-K\partial}(z), q^{-K\partial}(w))}{q(z, w)}, \quad (16)$$

$$(K^+(z), e(w)) = q(z, w), \quad (K^-(z), e(w)) = q(w, q^{-K\partial}(z))^{-1}, \quad (17)$$

$$(K^+(z), f(w)) = q(z, w)^{-1}, \quad (K^-(z), f(w)) = q(w, z). \quad (18)$$

Set

$$k^+(z) = q^{(T_++U_+)h^+(z)}, \quad k^-(z) = \lambda(z)q^{\frac{1}{1+q^{-\partial}}h^-(z)},$$

with  $\lambda(z)$  the function such that

$$\lambda(z)\lambda(q^{-\partial}z)q^{[\frac{1}{1+q^{-\partial}}h^-(z), \frac{q^{-\partial}}{1+q^{-\partial}}h^-(z)]} = 1,$$

that is

$$\lambda(z) = \exp \left[ -\frac{1}{1+q^{-\partial}z} \left( \left( \frac{1}{1+q^{-\partial}z} \otimes \frac{q^{-\partial_{z'}}}{1+q^{-\partial_{z'}}} \right) [h^-(z), h^-(z')] \right)_{z'=z} \right].$$

We have

$$K^+(z) = k^+(z)k^+(q^\partial z), \quad K^-(z) = k^-(z)^{-1}k^-(q^{-\partial}z)^{-1}.$$

Set

$$q_+(z, w) = q^{2\sum_i (T_++U_+)e_i(z) \otimes e^i(w)}, \quad q_-(z, w) = q^{-2\sum_i \frac{1}{1+q^{-\partial}}e^i(z) \otimes (T+U)e_i(w)},$$

then we have  $q_+(z, w) = q_-(z, w)$  (up to analytic continuation). We have

$$q_+(z, w)q_+(q^\partial z, w) = q(z, w),$$

and

$$\begin{aligned}(k^+(z), e(w)) &= q_+(z, w), & (k^-(z)^{-1}, e(w)) &= q_-(q^{(-K+1)\partial}z, w), \\ (k^+(z), f(w)) &= q_+(z, w)^{-1}, & (k^-(z)^{-1}, f(w)) &= q_-(q^\partial z, w)^{-1}.\end{aligned}$$

Also when  $K = -2$ , we have

$$(k^+(z), k^+(w)) = 1, \quad (k^+(z), k^-(w)) = \frac{q_+(z, q^\partial w)}{q_+(z, q^{2\partial} w)}, \quad (19)$$

and

$$(k^-(z), k^-(w)) = \frac{q_+(q^{3\partial} z, q^{2\partial} w)}{q_+(q^\partial z, q^\partial w)} \frac{q_-(q^{2\partial} w, q^{2\partial} z)}{q_-(q^{2\partial} w, q^\partial z)}. \quad (20)$$

### 3. SUBALGEBRA $U_\hbar \mathfrak{b}_{in}$

The quantity

$$(q^{\partial_z + \partial_w} - 1) \sum_i ((T + U)e_i)(z) e^i(w)$$

belongs to  $(R \otimes R)[[\hbar]]$ , since  $Ue_i$  belongs to  $R$ ,  $T$  commutes with  $\partial_z + \partial_w$  and  $(\partial_z + \partial_w)G(w, z) = -\sum_i e^i(z)(\partial e_i)_R(w)$  belongs to  $R \otimes R$ . Moreover

$$F(z, w) = 2\hbar \frac{q^{\partial_z + \partial_w} - 1}{(1 + q^{-\partial_z})(1 + q^{-\partial_w})} \sum_i (T + U)e_i(z) e^i(w)$$

is symmetric in  $z$  and  $w$ . Let  $\alpha(z, w)$  be an element of  $\hbar(R \otimes R)[[\hbar]]$  such that

$$\frac{\exp(2\alpha(q^\partial w, z))}{\exp(2\alpha(q^\partial z, w))} = \exp[2\hbar \frac{q^{\partial_z + \partial_w} - 1}{(1 + q^{-\partial_z})(1 + q^{-\partial_w})} \sum_i (T + U)e_i(z) e^i(w)]; \quad (21)$$

we may choose

$$\alpha(z, w) = \frac{1}{2} F(w, q^{-\partial} z).$$

Let us set

$$\alpha(z, w) = \sum_{i,j} a_{ij} e^i(z) e^j(w),$$

and

$$k_R(z) = \exp(\sum_{i,j} a_{ij} \hbar [e^i] e^j(z)).$$

Define  $R_{(a)}$  as  $\oplus_a \mathbb{C} r_a \oplus R$ . Recall that we defined  $\mathfrak{m}$  as  $\mathfrak{m}_1 \oplus \mathcal{O}_2 \oplus \cdots \oplus \mathcal{O}_p$ , so that  $\mathcal{K} = R_{(a)} \oplus \mathfrak{m}$ . Let  $\mathcal{A}$  be the  $\mathbb{C}[[\hbar]]$ -module automorphism of  $R_{(a)}[[\hbar]]$  defined by

$$\mathcal{A}(r) = r \text{ for } r \text{ in } R, \text{ and } \mathcal{A}(r_a) = (T + U)r_a \text{ for } a = 1, \dots, g. \quad (22)$$

Define  $\beta(z, w)$  in  $(R \otimes R_{(a)})[[\hbar]]$  by

$$\begin{aligned} & - 2(1 \otimes q^\partial \mathcal{A})\beta(z, w) \\ & = 2(\alpha(q^{2\partial} z, w) - \alpha(q^\partial z, w)) - 2\hbar \sum_i \frac{1}{1 + q^{-\partial}} e^i(z) \otimes ((T + U)e_i)_{R_{(a)}}(w) \\ & + 2\hbar(q^{3\partial} \otimes q^\partial - q^\partial \otimes q^{-\partial}) \left( \frac{1}{1 + q^\partial} \otimes \frac{1}{1 + q^{-\partial}} \right) \sum_i ((T + U)e_i)(z) \otimes e^i(w). \end{aligned}$$

Set

$$\beta(z, w) = \sum_{a,i} b_{ai} e^i(z) r_a(w) + \sum_{i,j} c_{ij} e^j(z) e^i(w)$$

and

$$k_a(z) = \exp\left(\sum_{a,i} b_{ai} \hbar [r_a] e^i(z) + \sum_{i,j} c_{ij} \hbar [e^i] e^j(z)\right). \quad (23)$$

Set finally

$$k_{\mathfrak{m}}(z) = k_a(z)^{-1} k^-(z).$$

The currents  $k_R(z)$ ,  $k_a(z)$  and  $k_{\mathfrak{m}}(z)$  all belong to  $U_{\hbar} \mathfrak{h} \otimes R_z[[\hbar]]$ .

**Proposition 3.1.** *i) Set  $\tilde{f}(z) = f(z)k_R(z)k^-(q^{-\partial}z)$ . we have  $\tilde{f}(z)\tilde{f}(w) = \tilde{f}(w)\tilde{f}(z)$ .  
ii) We have*

$$(k_{\mathfrak{m}}(z), \tilde{f}(w)) \in \exp(\hbar(R \otimes \mathfrak{m})[[\hbar]]).$$

*iii) We have  $k_{\mathfrak{m}}(z)k_{\mathfrak{m}}(w) = k_{\mathfrak{m}}(w)k_{\mathfrak{m}}(z)$ .*

*Proof.* Let us show that  $\tilde{f}(z)$  commutes with itself. We have

$$[h[r], f(z)k^-(q^{-\partial}z)] = -2(q^\partial r)(z)f(z)k^-(q^{-\partial}z),$$

therefore

$$\frac{(k_R(w), f(z)k^-(q^{-\partial}z))}{(k_R(z), f(w)k^-(q^{-\partial}w))} = \frac{\exp(2\alpha(q^\partial z, w))}{\exp(2\alpha(q^\partial w, z))}.$$

Set

$$j(z, w) = q_+(q^\partial z, w)q_-(w, z), \quad (24)$$

we have  $j(z, w) \in 1 + \hbar(R \otimes R)[[\hbar]]$ ,  $j(z, w)j(w, z) = 1$ . From [8] follows that for some  $i_-(z, w)$  in  $\mathbb{C}[[z, w]][z^{-1}, w^{-1}][[\hbar]]^\times$ , we have

$$q_-(z, w) = i_-(z, w) \frac{w - q^{-\partial}z}{w - z},$$

so that

$$j(z, w) = i_-(q^\partial z, w)i_-(w, z) \frac{z - q^{-\partial}w}{q^\partial z - w}.$$

We have then

$$\begin{aligned}
& (w - z)f(z)k^-(q^{-\partial}z)f(w)k^-(q^{-\partial}w) \\
&= i_-(z, w)(w - q^{-\partial}z)f(z)f(w)k^-(q^{-\partial}z)k^-(q^{-\partial}w) \\
&= (k^-(q^{-\partial}z), k^-(q^{-\partial}w))i_-(q^\partial z, w)^{-1}(w - q^\partial z)f(w)f(z)k^-(q^{-\partial}w)k^-(q^{-\partial}z) \\
&= (k^-(q^{-\partial}z), k^-(q^{-\partial}w))i_-(q^\partial z, w)^{-1}(w - z)\frac{w - q^\partial z}{q^{-\partial}w - z}f(w)k^-(q^{-\partial}w)f(z)k^-(q^{-\partial}z),
\end{aligned}$$

therefore

$$\begin{aligned}
& (w - z) \left[ (f(z)k^-(q^{-\partial}z)) (f(w)k^-(q^{-\partial}w)) \right. \\
& \quad \left. - \frac{(k^-(q^{-\partial}z), k^-(q^{-\partial}w))}{j(z, w)} (f(w)k^-(q^{-\partial}w)) (f(z)k^-(q^{-\partial}z)) \right] = 0;
\end{aligned}$$

let  $B(z, w)$  be the term in brackets. It is equal to  $A(z)\delta(z, w)$ , for some generating series  $A(z)$ . Since  $B(z, w)$  also satisfies  $B(w, z) = -\frac{(k^-(q^{-\partial}w), k^-(q^{-\partial}z))}{j(w, z)}B(z, w)$ , and  $\frac{(k^-(q^{-\partial}w), k^-(q^{-\partial}z))}{j(w, z)} = 1 + o(\hbar)$ , we obtain that  $B(z, w)$  vanishes so

$$\begin{aligned}
& (f(z)k^-(q^{-\partial}z)) (f(w)k^-(q^{-\partial}w)) \\
&= \frac{(k^-(q^{-\partial}z), k^-(q^{-\partial}w))}{j(z, w)} (f(w)k^-(q^{-\partial}w)) (f(z)k^-(q^{-\partial}z)).
\end{aligned} \tag{25}$$

We have

$$(k^-(z), k^-(w)) = \exp[2\hbar(q^{2(\partial_z + \partial_w)} - 1)] \frac{1}{1 + q^{-\partial_z}} \frac{1}{1 + q^{-\partial_w}} \sum_i (T + U)e_i(z)e^i(w),$$

$$j(z, w) = \exp[2\hbar(1 - q^{-\partial_z - \partial_w})] \frac{1}{1 + q^{-\partial_z}} \frac{1}{1 + q^{-\partial_w}} \sum_i (T + U)e_i(z)e^i(w), \tag{26}$$

so

$$\frac{(k^-(q^{-\partial}z), k^-(q^{-\partial}w))}{j(z, w)} = \exp[2\hbar \frac{q^{\partial_z + \partial_w} - 1}{(1 + q^{-\partial_z})(1 + q^{-\partial_w})} \sum_i (T + U)e_i(z)e^i(w)];$$

since  $k_R(z)$  commutes with  $k_R(w)$ , and by (25), *i*) follows.

Let us prove *ii*). We have

$$(k^-(z), \tilde{f}(w)) = \exp[2(\alpha(q^{2\partial}z, w) - \alpha(q^\partial z, w))]q_-(q^\partial z, w)(k^-(z), k^-(q^{-\partial}w));$$

moreover,

$$q_-(q^\partial z, w) = \exp[-2\hbar \sum_i \frac{1}{1 + q^{-\partial}} e^i(z) \otimes (T + U)e_i(w)],$$

From (23) follows that

$$(k_a(z), \tilde{f}(w)) = \exp[2(\alpha(q^{2\partial}z, w) - \alpha(q^\partial z, w))] \cdot \\ \cdot \exp[-2\hbar \sum_i \frac{1}{1+q^{-\partial}} e^i(z) \otimes ((T+U)e_i)_{R(a)}(w)](k^-(z), k^-(q^{-\partial}w)).$$

Therefore,

$$(k_m(z), \tilde{f}(w)) = \exp[-2\hbar \sum_i \frac{1}{1+q^{-\partial}} e^i(z) \otimes ((T+U)e_i)_m(w)], \quad (27)$$

which implies *ii*).

Set for  $\phi = \sum_a \lambda_a r_a + r$ , with  $\lambda_a$  in  $\mathbb{C}$  and  $r$  in  $R$ ,  $h[\phi] = \sum_a \lambda_a h^-[r_a] + h^+[r]$ . Then we have for  $\phi$  in  $R_{(a)}$ ,

$$[h[\phi], f(z)] = -2(\mathcal{A}\phi)(z)f(z)$$

and

$$[h[\phi], k^-(z)] = 2[q^\partial(1 - q^\partial)\mathcal{A}\phi](z)k^-(z),$$

where  $\mathcal{A}$  is defined by (22), so that

$$[h[\phi], \tilde{f}(z)] = -2[q^\partial\mathcal{A}\phi](z)\tilde{f}(z).$$

Therefore we get

$$(k_a(z), k^-(w)) = \frac{(k_a(z), \tilde{f}(q^\partial w))}{(k_a(z), \tilde{f}(w))} \\ = \exp[2(q^{2\partial_z + \partial_w} - q^{\partial_z + \partial_w} - q^{2\partial_z} + q^{\partial_z})\alpha(z, w)] \cdot \\ \cdot \exp[-2\hbar \sum_i \frac{1}{1+q^{-\partial}} e^i(z) \otimes (q^\partial - 1)((T+U)e_i)_{R(a)}(w)] \cdot \\ \cdot \frac{(k^-(z), k^-(w))}{(k^-(z), k^-(q^{-\partial}w))}.$$

On the other hand, *iii*) is translated as

$$\frac{(k_a(z)^{-1}, k^-(w))}{(k_a(w)^{-1}, k^-(z))}(k^-(z), k^-(w)) = 1,$$

that is

$$\exp[(q^{\partial_z} - 1)(q^{\partial_w} - 1)(2\alpha(q^\partial z, w) - 2\alpha(q^\partial w, z))] \\ \exp[-2\hbar \sum_i \frac{1}{1+q^{-\partial}} e^i(z) \otimes (q^\partial - 1)((T+U)e_i)_{R(a)}(w)] : (z \leftrightarrow w) \\ \frac{(k^-(z), k^-(w))^2}{(k^-(z), k^-(q^{-\partial}w))(k^-(q^{-\partial}z), k^-(w))} \\ = (k^-(z), k^-(w)),$$

in other terms

$$\begin{aligned}
& \exp[(q^{\partial_z} - 1)(q^{\partial_w} - 1) \log \left( \frac{(k^-(q^{-\partial}z), k^-(q^{-\partial}w))}{j(z, w)} \right)^{-1}] \\
& \exp[-2\hbar \sum_i \frac{1}{1 + q^{-\partial}} e^i(z) \otimes (q^\partial - 1)((T + U)e_i)_{R(a)}(w)] : (z \leftrightarrow w) \\
& \frac{(k^-(z), k^-(w))}{(k^-(z), k^-(q^{-\partial}w))(k^-(q^{-\partial}z), k^-(w))} \\
& = 1,
\end{aligned}$$

or

$$\begin{aligned}
& \exp[-2\hbar \sum_i \frac{1}{1 + q^{-\partial}} e^i \otimes (q^\partial - 1)((T + U)e_i)_{R(a)}] : (z \leftrightarrow w) \quad (28) \\
& \exp[(q^{\partial_z} - 1)(q^{\partial_w} - 1) \log j(z, w)] \\
& = (k^-(q^{-\partial}z), k^-(q^{-\partial}w)).
\end{aligned}$$

The terms containing  $U$  in the logarithm of (28) are

$$\begin{aligned}
& -2\hbar \left( \frac{1}{1 + q^{-\partial}} \otimes (q^\partial - 1) \right) \sum_i e^i \otimes Ue_i \\
& + 2\hbar \left( (q^\partial - 1) \otimes \frac{1}{1 + q^{-\partial}} \right) \sum_i Ue_i \otimes e^i + 2\hbar \left( \frac{q^\partial - 1}{1 + q^{-\partial}} \otimes (q^\partial - 1) \right) \sum_i Ue_i \otimes e^i \\
& - 2\hbar \left( (q^\partial - 1) \otimes \frac{q^\partial - 1}{q^\partial + 1} \right) \sum_i Ue_i \otimes e^i \\
& - 2\hbar (q^\partial \otimes q^\partial - q^{-\partial} \otimes q^{-\partial}) \left( \frac{1}{1 + q^{-\partial}} \otimes \frac{1}{1 + q^{-\partial}} \right) \left( \sum_i Ue_i \otimes e^i \right)
\end{aligned}$$

which is equal to

$$2\hbar \left( \frac{1}{1 + q^{-\partial}} \otimes (q^\partial - 1) \right) \sum_i e^i \otimes (Te_i)_R,$$

in view of (5).

Therefore *iii)* is written as

$$\begin{aligned}
& \exp[-2\hbar(\frac{1}{1+q^{-\partial}} \otimes (q^\partial - 1)) \sum_i e^i \otimes (Te_i)_{R(a)}] : (z \leftrightarrow w) \\
& \exp[2\hbar(\frac{1}{1+q^{-\partial}} \otimes (q^\partial - 1)) \sum_i e^i \otimes (Te_i)_R] \\
& \exp[2\hbar(\frac{q^\partial - 1}{1+q^{-\partial}} \otimes (q^\partial - 1)) \sum_i Te_i \otimes e^i] \exp[-2\hbar((q^\partial - 1) \otimes \frac{q^\partial - 1}{1+q^\partial}) \sum_i Te_i \otimes e^i] \\
& = \exp[2\hbar(q^\partial \otimes q^\partial - q^{-\partial} \otimes q^{-\partial})(\frac{1}{1+q^{-\partial}} \otimes \frac{1}{1+q^{-\partial}}) \sum_i Te_i \otimes e^i]
\end{aligned}$$

or

$$\begin{aligned}
& \exp[2\hbar(\frac{1}{1+q^{-\partial}} \otimes (1 - q^\partial))(\sum_i Te_i \otimes e^i + e^i \otimes (Te_i)_{R(a)} - e^i \otimes (Te_i)_R)] \\
& \exp[2\hbar((1 - q^\partial) \otimes \frac{1}{1+q^{-\partial}})(\sum_i Te_i \otimes e^i - (Te_i)_{R(a)} \otimes e^i)] = 1. \tag{29}
\end{aligned}$$

Since

$$\sum_i Te_i \otimes e^i + e^i \otimes (Te_i)_{R(a)} - e^i \otimes (Te_i)_R$$

is

$$(T \otimes id)\delta(z, w) - \left( \sum_i Te_i \otimes e^i - (Te_i)_{R(a)} \otimes e^i \right)^{(21)}, \tag{30}$$

(29) is equivalent to the statement that

$$\begin{aligned}
& ((1 - q^\partial) \otimes \frac{1}{1+q^{-\partial}}) \left( \sum_i Te_i \otimes e^i - (Te_i)_{R(a)} \otimes e^i \right) - (z \leftrightarrow w) \\
& + \left( \frac{(q^\partial - 1)(1 - q^{-\partial})}{2\hbar\partial} \otimes id \right) \delta(z, w) = 0
\end{aligned} \tag{31}$$

(whose interpretation is that after analytic prolongation,

$$((1 - q^\partial) \otimes \frac{1}{1+q^{-\partial}}) \left( \sum_i Te_i \otimes e^i - (Te_i)_{R(a)} \otimes e^i \right)$$

is symmetric in  $z$  and  $w$ ).

To prove this, we first show that

**Lemma 3.1.** *One can choose the dual bases  $(e_i)_{i \geq 0}$ ,  $(e^i)_{i \geq 0}$  as  $(r_a; e'_i)_{a=1, \dots, g, i \geq 0}$  and  $(\omega_a/\omega; e^i)_{a=1, \dots, g, i \geq 0}$ , with  $(e'_i)_{i \geq 0}$  be a basis of  $\mathfrak{m}$  and  $(e^i)_{i \geq 0}$  the dual basis of*

the subspace  $K_a$  of  $R$  defined as  $\{r \in R \mid \int_{A_i} r \omega = 0\}$ . We have

$$\sum_{i \geq 0} (Te_i)_{\mathfrak{m}} \otimes e^i = \sum_{i \geq 0} e'_i \otimes Te^i. \quad (32)$$

*Proof of Lemma.* The first statement follows from the fact that  $K_a$  is the annihilator of  $\oplus_a \mathbb{C}r_a$  in  $R$  for  $\langle, \rangle_{\mathcal{K}}$ . Let us show the second statement. Both sides of the equality belong to  $\mathfrak{m} \otimes \mathcal{K}$ . On the other hand, the annihilator of  $K_a$  for  $\langle, \rangle_{\mathcal{K}}$  is  $R_{(a)}$  and has therefore zero intersection with  $\mathfrak{m}$ . It follows that to show (32), it is enough to show that the pairing of both sides with  $\rho \otimes id$  coincide, for  $\rho$  in  $K_a$ . But  $\langle \text{left side}, \rho \otimes id \rangle = \sum_i (Te_i)_{\mathfrak{m}}, \rho \rangle e^i = \sum_i \langle Te_i, \rho \rangle e^i$ , because  $K_a$  and  $R_{(a)}$  are orthogonal; this is equal to  $\sum_i \langle e_i, T\rho \rangle e^i$  because  $T$  is self-adjoint and therefore to  $T\rho$ . On the other hand,  $\langle \text{right side}, \rho \otimes id \rangle = \sum_i \langle e'_i, \rho \rangle Te^i = T\rho$ . This proves (32).  $\square$

(31) is equal to

$$((1 - q^\partial) \otimes \frac{1}{1 + q^{-\partial}}) \sum_i (Te_i)_{\mathfrak{m}} \otimes e^i;$$

by Lemma 3.1, this is

$$((1 - q^\partial) \otimes \frac{1}{1 + q^{-\partial}}) \sum_i e'_i \otimes Te^i,$$

which is

$$(1 - q^\partial) \otimes \frac{q^\partial - 1}{2\hbar\partial} \sum_i e'_i \otimes e^i$$

or

$$-\frac{1}{2\hbar} \left( \frac{q^\partial - 1}{\partial} \otimes \frac{q^\partial - 1}{\partial} \right) \sum_i \partial e'_i \otimes e^i.$$

(31) now follows from

$$\sum_i \partial e'_i \otimes e^i - \sum_i e^i \otimes \partial e'_i = (\partial \otimes id) \delta(z, w)$$

(which means that after analytic continuation,  $\sum_i \partial e'_i \otimes e^i$  is symmetric). This equality can be proved either by expressing  $\sum_i e'_i \otimes e^i$  explicitly using theta-functions (see [4]), or as follows:  $\sum_i \partial e'_i \otimes e^i - e^i \otimes \partial e'_i - (\partial \otimes id) \delta(z, w)$  is equal to

$$\sum_i \partial e'_i \otimes e^i + e'_i \otimes \partial e^i + \sum_a \omega_a / \omega \otimes \partial r_a, \quad (33)$$

which belongs to  $\mathcal{K} \otimes R$ . To show that (33) is zero, let us pair in with  $id \otimes \lambda$ ,  $\lambda$  in  $\mathfrak{m} \oplus \oplus_a \mathbb{C}r_a$ . For  $o$  in  $\mathfrak{m}$ ,  $\langle (33), id \otimes o \rangle$  is equal to

$$(\partial o)_{R_{(a)} \parallel \mathfrak{m}} - \sum_a \omega_a / \omega \langle o, \partial r_a \rangle;$$

but  $\partial o$  belongs to  $\mathfrak{m} \oplus (\oplus_a \mathbb{C} r_a) \oplus (\oplus_a \mathbb{C} \omega_a / \omega)$ , therefore this vanishes. On the other hand,  $\langle (33), id \otimes r_a \rangle$  is equal to zero, because  $\langle \partial r_a, r_b \rangle = 0$  for any  $a, b$ .  $\square$

From the proof of Prop. 3.1 follows that  $(k_{\mathfrak{m}}(z), \tilde{f}(w))$  is of the form  $\exp(-2\hbar \sum_i e'^i \otimes e'_i + o(\hbar))$ , so that we have

$$(k_{\mathfrak{m}}(z), \tilde{f}(w)) = \exp(-2\hbar \sum_{i \geq 0} \varphi^i \otimes e'_i), \quad (34)$$

with  $(\varphi^i)_{i \geq 0}$  a free family of  $R[[\hbar]]$ . Set  $h_{\mathfrak{m}}(z) = \frac{1}{\hbar} \ln k_{\mathfrak{m}}(z)$ ; (34) implies that

$$h_{\mathfrak{m}}(z) = \sum_{i \geq 0} \tilde{h}[e'_i] \varphi^i(z),$$

with  $\tilde{h}[e'_i]$  linear combinations of the  $h^+[r]$  and  $h^-[\lambda]$ . Define  $\tilde{h}[o]$  for  $o$  in  $\mathfrak{m}$  by linear extension.

**Corollary 3.1.** *Define in  $U_{\hbar, \omega} \mathfrak{g}$ ,  $\tilde{f}[\epsilon]$  as  $\sum_i \text{res}_{P_i}(\tilde{f}(z) \epsilon(z) \omega(z))$ . Let  $\mathfrak{b}_{in}$  be the Lie algebra*

$$\mathfrak{b}_{in} = (\bar{\mathfrak{h}} \otimes \mathfrak{m}) \oplus (\bar{\mathfrak{n}}_+ \otimes \mathcal{K}).$$

*Then there is an algebra injection  $U \mathfrak{b}_{in}[[\hbar]] \rightarrow U_{\hbar, \omega} \mathfrak{g}$  defined by  $h \otimes o \mapsto \tilde{h}[o]$ ,  $f \otimes \epsilon \mapsto \tilde{f}[\epsilon]$ . We define  $U_{\hbar} \mathfrak{b}_{in}$  as the image of this injection.*

*Proof.* From the construction of  $\tilde{h}[o]$  follows that we have

$$[\tilde{h}[o], \tilde{f}(z)] = -2o(z) \tilde{f}(z);$$

Prop. 3.1, *i*) and *iii*) then imply the statement.  $\square$

Define  $U_{\hbar} \mathfrak{b}_-$  as the subalgebra of  $U_{\hbar, \omega} \mathfrak{g}$  generated by the  $h^+[r]$ ,  $h^-[\lambda]$  and the  $\tilde{f}[\epsilon]$ ,  $r$  in  $R$ ,  $\lambda$  in  $\Lambda$ ,  $\epsilon$  in  $\mathcal{K}$ ;  $U_{\hbar} \mathfrak{b}_{in}$  is then a subalgebra of  $U_{\hbar} \mathfrak{b}_-$ . Define  $U_{\hbar} \mathfrak{b}_{\lambda_0}^{out}$  as the subalgebra of  $U_{\hbar} \mathfrak{b}_-$  generated by the  $h^+[r]$  and the  $\tilde{f}[r_{-2\lambda_0}]$ ,  $r$  in  $R$ ,  $r_{-2\lambda_0}$  in  $R_{-2\lambda_0}$ .

We have:

**Proposition 3.2.**  *$U_{\hbar} \mathfrak{b}_-$  is the direct sum of  $\mathbb{C}[h[r_a]] U_{\hbar} \mathfrak{b}^{in}$  and of its right ideal generated by its right ideal  $\sum_{r \in R} h^+[r] U_{\hbar} \mathfrak{b}_{\lambda_0}^{out} + \sum_{r_{-2\lambda_0} \in R_{-2\lambda_0}} \tilde{f}[r_{-2\lambda_0}] U_{\hbar} \mathfrak{b}_{\lambda_0}^{out}$ .*

*Proof.* For  $\rho$  in  $R_{(a)}$ , set  $\tilde{h}[\rho] = h[(q^\partial \mathcal{A})^{-1} \rho]$ . Extend  $\tilde{h}$  to  $\mathcal{K}$  by linearity. A system of relations for  $U_{\hbar} \mathfrak{b}_-$  is then

$$[\tilde{f}[\epsilon], \tilde{f}[\epsilon']] = 0, \quad [\tilde{h}[\epsilon], \tilde{f}[\eta]] = -2\tilde{f}[\epsilon\eta],$$

$$[\tilde{h}[\epsilon], \tilde{h}[\epsilon']] = f(\epsilon, \epsilon'),$$

with  $f(\epsilon, \epsilon')$  scalar, for  $\epsilon, \epsilon', \eta$  in  $\mathcal{K}$ . Denote by  $\mathbb{C}\langle \phi, \phi \in F \rangle$  the subalgebra of  $U_{\hbar} \mathfrak{b}_-$  generated by the family  $F$  of elements of  $U_{\hbar} \mathfrak{b}_-$ . The product map from

$$\mathbb{C}\langle \tilde{h}[o], \tilde{f}[\lambda'], o \in \mathfrak{m}, \lambda' \in \Lambda \rangle \otimes \mathbb{C}\langle \tilde{h}[r_a] \rangle \otimes \mathbb{C}\langle \tilde{h}[r], \tilde{f}[r_{-2\lambda_0}], r \in R, r_{-2\lambda_0} \in R_{-2\lambda_0} \rangle$$

to  $U_{\hbar}\mathfrak{b}_-$  then defines an isomorphism. Therefore  $U_{\hbar}\mathfrak{b}_-$  is the direct sum of  $\mathbb{C}[\hbar[r_a]]U_{\hbar}\mathfrak{b}_{\lambda_0}^{out}$  and the left ideal  $I$  generated by the  $\tilde{h}[r], \tilde{f}[r_{-2\lambda_0}]$ ,  $r$  in  $R$ ,  $r_{-2\lambda_0}$  in  $R_{-2\lambda_0}$ .  $\mathbb{C}[\hbar[r_a]]U_{\hbar}\mathfrak{b}_{\lambda_0}^{out}$  is equal to  $\mathbb{C}[\hbar[r_a]]U_{\hbar}\mathfrak{b}_{\lambda_0}^{out}$ ; on the other hand,  $\tilde{f}[r_{-2\lambda_0}] = \sum_i \text{res}_{P_i}(f(z)k^-(q^{-\partial}z)r_{-2\lambda_0}(z)\omega_z)$ , and  $k^-(q^{-\partial}z)$  belongs to  $U_{\hbar,\omega}\mathfrak{g} \otimes R_z$ , so that since  $R_{-2\lambda_0}$  is a  $R$ -module,  $\tilde{f}[r_{-2\lambda_0}]$  belongs to  $f[r_{-2\lambda_0}] + \hbar(\text{right ideal generated by the } f[\rho_{-2\lambda_0}], \rho_{-2\lambda_0} \text{ in } R_{-2\lambda_0})$ . Therefore,  $I$  coincides with the left ideal generated by the  $\hbar[r], f[r_{-2\lambda_0}]$ ,  $r$  in  $R$ ,  $r_{-2\lambda_0}$  in  $R_{-2\lambda_0}$ , which is the augmentation ideal of  $U_{\hbar}\mathfrak{b}_{\lambda_0}^{out}$ . The Lemma follows.  $\square$

#### 4. A PRESENTATION OF $U_{\hbar,\omega}\mathfrak{g}/(K+2)$

Set

$$\tilde{e}(z) = k^+(q^{\partial}z)^{-1}k_R(z)^{-1}e(q^{\partial}z),$$

an

$$k_{tot}^+(z) = k^+(q^{2\partial}z)k_R(q^{\partial}z)k_R(z)^{-1}k^-(z), \quad (35)$$

$$k_{tot}^-(z) = k^+(q^{\partial}z)^{-1}k_R(z)^{-1}k_R(q^{-\partial}z)k^-(q^{-\partial}z)^{-1}. \quad (36)$$

**Proposition 4.1.** *The following relations*

$$[\tilde{e}(z), \tilde{e}(w)] = [\tilde{f}(z), \tilde{f}(w)] = 0, \quad (37)$$

$$[\tilde{e}(z), \tilde{f}(w)] = \frac{1}{\hbar}\delta(q^{\partial}z, w)k_{tot}^+(z) - \frac{1}{\hbar}\delta(z, q^{\partial}w)k_{tot}^-(z) \frac{\exp(2\alpha(q^{-\partial}z, q^{-\partial}z))}{\exp(2\alpha(q^{\partial}z, q^{-\partial}z))} \quad (38)$$

are satisfied in  $U_{\hbar,\omega}\mathfrak{g}$ .

*Proof.* The proof of  $[\tilde{e}(z), \tilde{e}(w)] = 0$  is similar to that of Prop. 3.1, *i*) and relies on the identities  $(k_R(z), e(q^{\partial}w)) = \exp(2\alpha(q^{\partial}w, z))$ , and

$$\frac{\exp(2\alpha(q^{\partial}z, w))}{\exp(2\alpha(q^{\partial}w, z))}j(q^{\partial}z, q^{\partial}w) = 1, \quad (39)$$

which follows from (21) and (26).

Then we have

$$\begin{aligned} \tilde{f}(w)\tilde{e}(z) &= (k_R(w)k^-(q^{-\partial}w), k^+(q^{\partial}z)^{-1}k_R(z)^{-1}) \\ &\quad (f(w), k^+(q^{\partial}z)^{-1}k_R(z)^{-1})(k_R(w)k^-(q^{-\partial}w), e(z)) \\ &\quad k^+(q^{\partial}z)^{-1}k_R(z)^{-1}f(w)e(q^{\partial}z)k_R(w)k^-(q^{-\partial}w); \end{aligned}$$

this equation may be written as

$$\tilde{f}_n\tilde{e}_m = \sum_{p \geq 0} \hbar^p \sum_{i \geq N(p), j \geq M(p)} A_{ij}^{(p)} k_{-i}^+ f_{n-j} e_{m+i} k_j^-,$$

where we set  $x(z) = \sum_n x_n z^{-n}$ ,  $x = e, f, \tilde{e}, \tilde{f}, k^\pm$ ; the right side belongs to the completion  $U_{\hbar, \omega} \mathfrak{g}$ . Equation (38) then follows from the identity

$$(K^-(q^{-\partial} z), k_R(q^{-\partial} z)) = \frac{\exp(2\alpha(q^{-\partial} z, q^{-\partial} z))}{\exp(2\alpha(q^{\partial} z, q^{-\partial} z))}.$$

□

**Theorem 4.1.**  $U_{\hbar, \omega} \mathfrak{g}/(K+2)$  has a presentation with generating series  $\tilde{e}(z), \tilde{f}(z), k^\pm(z)$  and relations (19), (20), (35), (36), (37), (38), and

$$(k^+(z), \tilde{e}(w)) = q_+(z, q^{\partial} w), \quad (40)$$

$$(k^-(z), \tilde{e}(w)) = \exp[2\alpha(q^{\partial} z, w) - 2\alpha(q^{2\partial} z, w)] \frac{q_+(q^{\partial} w, q^{\partial} z)}{q_+(q^{\partial} w, q^{2\partial} z) q_-(q^{3\partial} z, q^{\partial} w)} \quad (41)$$

and

$$(k^+(z), \tilde{f}(w)) = q_+(z, q^{\partial} w)^{-1}, \quad (k^-(z), \tilde{f}(w)) = (k^-(z), \tilde{e}(w))^{-1}. \quad (42)$$

## 5. CENTRAL CURRENT $T(z)$

Recall that

$$g_\lambda^+(z) = (G_{-2\lambda} - G)(q^{\partial} z, z), \quad g_\lambda^-(z) = (G_{-2\lambda} - G)(q^{-\partial} z, z).$$

Define  $\sigma, \alpha, \beta$  in  $R[[\hbar]]$  and  $A_\lambda, B_\lambda$  in  $\mathcal{K}[[\hbar]]$  by

$$\sigma(q^{\partial} z) = \left[ -e^{-2\sum_i (q^{\partial} U_{+e_i})(z) \otimes e^i(w)} e^{-\phi(\hbar, \partial_z^i \gamma)} \psi(\hbar, \partial_z^i \gamma) \right]_{z=w}; \quad (43)$$

$$\alpha(q^{\partial} z) = \left[ -e^{-2\sum_i (q^{\partial} U_{+e_i})(z) \otimes e^i(w)} \partial_\hbar \{ e^{-\phi(\hbar, \partial_z^i \gamma)} \psi(\hbar, \partial_z^i \gamma) \} \right]_{w=z}, \quad (44)$$

$$\beta(q^{\partial} z) = \alpha(q^{\partial} z) - 2\partial_\hbar[\tau_{w=z}] \sigma(q^{\partial} z), \quad (45)$$

$$A_\lambda(z) = \alpha(q^{2\partial} z) + \sigma(q^{2\partial} z) [g_\lambda^+(z) - \sum_i e^i(z) (q^{2\partial} (q^{-\partial} e_i)_R)(z)], \quad (46)$$

$$B_\lambda(z) = \beta(q^{2\partial} z) - \sigma(q^{2\partial} z) [g_\lambda^-(z) - \sum_i e^i(z) ((q^{-\partial} e_i)_R)(z)], \quad (47)$$

we have  $\sigma(z) = \hbar + O(\hbar^2)$ ,  $\alpha(z) = 1 + O(\hbar)$ ,  $\beta(z) = 1 + O(\hbar)$ ,  $A_\lambda = 1 + O(\hbar)$ ,  $B_\lambda = 1 + O(\hbar)$ .

Let us set

$$T(z) = \tilde{e}(z) \tilde{f}(z)_{z \rightarrow R_{2\lambda}} + \tilde{f}(z)_{z \rightarrow \Lambda'} \tilde{e}(z) + a_\lambda(z) k_{tot}^+(z) + b_\lambda(z) k_{tot}^-(z),$$

where

$$a_\lambda(z) = \frac{1}{\hbar} \frac{A_\lambda(z)}{\sigma(q^{2\partial} z)}, \quad (48)$$

and

$$b_\lambda(z) = \frac{1}{\hbar} \frac{\exp(2\alpha(q^{-\partial}z, q^{-\partial}z))}{\exp(2\alpha(q^\partial z, q^\partial z))} \frac{B_\lambda(z)}{\sigma(q^{2\partial}z)}; \quad (49)$$

we will also set

$$b'_\lambda(z) = \frac{1}{\hbar} \frac{B_\lambda(z)}{\sigma(q^{2\partial}z)}. \quad (50)$$

**Theorem 5.1.** *The Laurent coefficients of  $T(z)$  are central elements of  $U_{\hbar, \omega} \mathfrak{g}$ .*

The proof is contained in the next sections.

**5.1. Commutation of  $T(z)$  with  $\tilde{e}(w)$ .** Set

$$: k_{tot}^+(z) \tilde{e}(w) := k^+(q^{2\partial}z) k^+(q^\partial w)^{-1} k_R(q^\partial z) k_R(z)^{-1} k_R(w)^{-1} \tilde{e}(q^\partial w) k^-(z),$$

and

$$: k_{tot}^-(z) \tilde{e}(w) := k^+(q^\partial z)^{-1} k_R(z)^{-1} k_R(q^{-\partial}z) k^+(q^\partial w)^{-1} k_R(w)^{-1} \tilde{e}(q^\partial w) k^-(q^{-\partial}z)^{-1}.$$

**Lemma 5.1.** *We have*

$$k_{tot}^+(z) \tilde{e}(w) = \exp(2\alpha(q^\partial w, z) - 2\alpha(q^\partial w, q^\partial z)) q_-(q^{2\partial}z, q^\partial w)^{-1} : k_{tot}^+(z) \tilde{e}(w) :,$$

and

$$\tilde{e}(w) k_{tot}^+(z) = \exp(2\alpha(q^\partial w, z) - 2\alpha(q^\partial w, q^\partial z)) q_+(q^{2\partial}z, q^\partial w)^{-1} : k_{tot}^+(z) \tilde{e}(w) :.$$

*Proof.* Let us prove the first identity. The factor in the right side is

$$(k^-(z), k^+(q^\partial w)^{-1})(k^-(z), k_R(w)^{-1})(k^-(z), e(q^\partial w)). \quad (51)$$

we have

$$(k^-(z), k_R(w)^{-1}) = \exp(2\alpha(q^\partial z, w) - 2\alpha(q^{2\partial}z, w)),$$

therefore (51) is equal to

$$\frac{q_+(q^\partial w, q^\partial z)}{q_+(q^\partial w, q^{2\partial}z)} \exp(2\alpha(q^\partial z, w) - 2\alpha(q^{2\partial}z, w)) q_-(q^{3\partial}z, q^\partial w)^{-1}. \quad (52)$$

Identity (39) can be formulated as

$$\frac{\exp(2\alpha(q^\partial w, z))}{\exp(2\alpha(q^\partial z, w))} = q_+(q^\partial w, q^\partial z) q_-(q^{2\partial}z, q^\partial w),$$

because the right side is  $j(q^\partial z, q^\partial w)$  (see (24)). Applying to this identity  $\exp \circ (1 - q^{\partial z}) \circ \log$ , we transform (52) into

$$\exp(2\alpha(q^\partial w, z) - 2\alpha(q^\partial w, q^\partial z)) q_-(q^{2\partial}z, q^\partial w)^{-1};$$

this implies the first equality.

The factor in the right side of the second identity is

$$(e(q^\partial w), k^+(q^{2\partial}z) k_R(q^\partial z) k_R(z)^{-1}),$$

which is equal to

$$\exp(2\alpha(q^\partial w, z) - 2\alpha(q^\partial w, q^\partial z))q_+(q^{2\partial}z, q^\partial w)^{-1}.$$

□

In the same way, one proves

**Lemma 5.2.** *We have*

$$k_{tot}^-(z)\tilde{e}(w) = \exp(2\alpha(q^\partial w, z) - 2\alpha(q^\partial w, q^{-\partial}z))q_-(q^\partial z, q^\partial w) : k_{tot}^-(z)\tilde{e}(w) :,$$

and

$$\tilde{e}(w)k_{tot}^-(z) = \exp(2\alpha(q^\partial w, z) - 2\alpha(q^\partial w, q^{-\partial}z))q_+(q^\partial z, q^\partial w) : k_{tot}^-(z)\tilde{e}(w) :.$$

Then we have

**Proposition 5.1.**  *$T(z)$  commutes with  $\tilde{e}(w)$ .*

*Proof.* We have

$$\begin{aligned} [T(z), \tilde{e}(w)] &= \tilde{e}(z)[\tilde{f}(z), \tilde{e}(w)]_{z \rightarrow R_{2\lambda}} + [\tilde{f}(z), \tilde{e}(w)]_{z \rightarrow \Lambda'} \tilde{e}(z) \\ &+ a_\lambda(z)[k_{tot}^+(z), \tilde{e}(w)] + b_\lambda(z)[k_{tot}^-(z), \tilde{e}(w)] \\ &= -\tilde{e}(z) \left( \frac{1}{\hbar} \delta(q^\partial w, z) k_{tot}^+(w) - \frac{1}{\hbar} \delta(w, q^\partial z) k_{tot}^-(w) \frac{\exp(2\alpha(q^{-\partial}w, q^{-\partial}w))}{\exp(2\alpha(q^\partial w, q^{-\partial}w))} \right)_{z \rightarrow R_{2\lambda}} \\ &- \left( \frac{1}{\hbar} \delta(q^\partial w, z) k_{tot}^+(w) - \frac{1}{\hbar} \delta(w, q^\partial z) k_{tot}^-(w) \frac{\exp(2\alpha(q^{-\partial}w, q^{-\partial}w))}{\exp(2\alpha(q^\partial w, q^{-\partial}w))} \right)_{z \rightarrow \Lambda'} \tilde{e}(z) \\ &+ a_\lambda(z)[k_{tot}^+(z), \tilde{e}(w)] + b_\lambda(z)[k_{tot}^-(z), \tilde{e}(w)] \\ &= -\frac{1}{\hbar} (G_{-2\lambda}(q^\partial w, z) k_{tot}^+(w) \tilde{e}(z) + G_{2\lambda}(z, q^\partial w) \tilde{e}(z) k_{tot}^+(w)) \\ &+ \frac{1}{\hbar} (G_{-2\lambda}(q^{-\partial}w, z) k_{tot}^-(w) \tilde{e}(z) + G_{2\lambda}(z, q^{-\partial}w) \tilde{e}(z) k_{tot}^-(w)) \frac{\exp(2\alpha(q^{-\partial}w, q^{-\partial}w))}{\exp(2\alpha(q^\partial w, q^{-\partial}w))} \\ &+ a_\lambda(z)[k_{tot}^+(z), \tilde{e}(w)] + b_\lambda(z)[k_{tot}^-(z), \tilde{e}(w)]; \end{aligned} \tag{53}$$

the last equality follows from the identities

$$\delta(w, z)_{z \rightarrow R_{2\lambda}} = G_{2\lambda}(z, w), \quad \delta(w, z)_{z \rightarrow \Lambda'} = G_{-2\lambda}(w, z).$$

We have

$$\begin{aligned} &G_{-2\lambda}(q^\partial w, z) k_{tot}^+(w) \tilde{e}(z) + G_{2\lambda}(z, q^\partial w) \tilde{e}(z) k_{tot}^+(w) \\ &= \exp(2\alpha(q^\partial z, w) - 2\alpha(q^\partial z, q^\partial w)) \\ &(G_{-2\lambda}(q^\partial w, z) q_-(q^{2\partial}w, q^\partial z)^{-1} + G_{2\lambda}(z, q^\partial w) q_+(q^{2\partial}w, q^\partial z)^{-1}) : k_{tot}^+(w) \tilde{e}(z) : \\ &= \exp(2\alpha(q^\partial z, w) - 2\alpha(q^\partial z, q^\partial w)) A_\lambda(z) \delta(z, w) : k_{tot}^+(w) \tilde{e}(z) :, \end{aligned}$$

where the first equality follows from Lemma 5.1, and the second from Lemma A.3.

In the same way, we have

$$\begin{aligned}
 & G_{-2\lambda}(q^{-\partial}w, z)k_{tot}^-(w)\tilde{e}(z) + G_{2\lambda}(z, q^{-\partial}w)\tilde{e}(z)k_{tot}^-(w) \\
 &= \exp(2\alpha(q^\partial z, w) - 2\alpha(q^\partial z, q^{-\partial}w)) \\
 & (G_{-2\lambda}(q^{-\partial}w, z)q_-(q^\partial w, q^\partial z) + G_{2\lambda}(z, q^{-\partial}w)q_+(q^\partial w, q^\partial z)) : k_{tot}^-(z)\tilde{e}(w) : \\
 &= \exp(2\alpha(q^\partial z, w) - 2\alpha(q^\partial z, q^{-\partial}w))B_\lambda(z)\delta(z, w) : k_{tot}^-(z)\tilde{e}(w) :
 \end{aligned}$$

where the first equality follows from Lemma 5.2, and the second from Lemma A.3.

On the other hand, we have

$$\begin{aligned}
 & [k_{tot}^+(z), \tilde{e}(w)] \\
 &= \exp(2\alpha(q^\partial w, z) - 2\alpha(q^\partial w, q^\partial z))[q_-(q^{2\partial}z, q^\partial w)^{-1} - q_+(q^{2\partial}z, q^\partial w)^{-1}] \\
 & : k_{tot}^+(z)\tilde{e}(w) : \\
 &= \exp(2\alpha(q^\partial w, z) - 2\alpha(q^\partial w, q^\partial z))\sigma(q^{2\partial}z)\delta(z, w) : k_{tot}^+(z)\tilde{e}(w) :
 \end{aligned}$$

and

$$\begin{aligned}
 & [k_{tot}^-(z), \tilde{e}(w)] \\
 &= \exp(2\alpha(q^\partial w, z) - 2\alpha(q^\partial w, q^{-\partial}z))[q_-(q^\partial w, q^\partial z) - q_+(q^\partial w, q^\partial z)] \\
 & : k_{tot}^-(z)\tilde{e}(w) : \\
 &= \exp(2\alpha(q^\partial w, z) - 2\alpha(q^\partial w, q^{-\partial}z))[-\sigma(q^{2\partial}w)\delta(z, w)] : k_{tot}^-(z)\tilde{e}(w) : .
 \end{aligned}$$

The equalities  $\delta(z, w) : k_{tot}^\pm(z)\tilde{e}(w) := \delta(z, w) : k_{tot}^\pm(w)\tilde{e}(z) :$  then imply that (53) vanishes.  $\square$

**5.2. Commutation of  $T(z)$  with  $k^\pm(w)$ .** Let us denote by  $U_{\hbar}\mathfrak{n}_+$ ,  $U_{\hbar}\mathfrak{h}$  and  $U_{\hbar}\mathfrak{n}_-$  the subalgebras of  $U_{\hbar, \omega}\mathfrak{g}$  generated respectively by the  $e[\epsilon]$ , by the  $h^+[r]$ ,  $h^-[\lambda]$  and  $K$ , and by the  $f[\epsilon]$ . If we assign degree 1 to the  $e[\epsilon]$  and  $f[\epsilon]$ ,  $U_{\hbar}\mathfrak{n}_\pm$  are graded algebras. We denote by  $U_{\hbar}\mathfrak{n}_\pm^{[i]}$  their homogeneous components of degree  $i$ .

We will prove

**Lemma 5.3.**  $k^+(w)T(z)k^+(w)^{-1} - T(z)$  and  $k^-(w)T(z)k^-(w)^{-1} - T(z)$  both belong to  $U_{\hbar}\mathfrak{h}$ .

*Proof.* It suffices to prove the same statements with  $T(z)$  replaced by  $T_0(z)$  defined by

$$T_0(z) = \tilde{e}(z)\tilde{f}(z)_{z \rightarrow R_{2\lambda}} + \tilde{f}(z)_{z \rightarrow \Lambda'}\tilde{e}(z).$$

Then from (40) and (42) follows that

$$\begin{aligned}
& q_+(z, q^\partial w)^{-1} k^+(w) T_0(z) k^+(w)^{-1} - T_0(z) \\
&= \tilde{e}(w) [q_+(z, q^\partial w)^{-1} \tilde{f}(w)]_{w \rightarrow R_{2\lambda}} + [q_+(z, q^\partial w)^{-1} \tilde{f}(w)]_{w \rightarrow \Lambda'} \tilde{e}(w) \\
&- q_+(z, q^\partial w)^{-1} [\tilde{e}(w) \tilde{f}(w)]_{w \rightarrow R_{2\lambda}} + \tilde{f}(w)_{w \rightarrow \Lambda'} \tilde{e}(w) \\
&= [\tilde{e}(w), [q_+(z, q^\partial w)^{-1} \tilde{f}(w)]_{w \rightarrow \Lambda'} - q_+(z, q^\partial w)^{-1} \tilde{f}(w)_{w \rightarrow \Lambda'}]
\end{aligned}$$

because of the identity

$$\begin{aligned}
& [q_+(z, q^\partial w)^{-1} \tilde{f}(w)]_{w \rightarrow \Lambda'} - q_+(z, q^\partial w)^{-1} \tilde{f}(w)_{w \rightarrow \Lambda'} \\
&= q_+(z, q^\partial w)^{-1} \tilde{f}(w)_{w \rightarrow R_{2\lambda}} - [q_+(z, q^\partial w)^{-1} \tilde{f}(w)]_{w \rightarrow R_{2\lambda}}.
\end{aligned}$$

For any  $\epsilon$  in  $\mathcal{K}$ ,  $[\tilde{e}[\epsilon], \tilde{f}(z)]$  belongs to  $U_{\hbar} \mathfrak{h}$ , which proves the first part of the statement. The second part is proved in the same way, using (40) and (42).  $\square$

Let us now prove

**Proposition 5.2.**  *$T(z)$  commutes with  $U_{\hbar} \mathfrak{h}$ .*

*Proof.* Set for  $r$  in  $R$  and  $\lambda$  in  $\Lambda$ ,

$$x_\eta^+(r) = [h^+[r], T[\eta]], \quad x_\eta^-(\lambda) = [h^-[\lambda], T[\eta]].$$

From Lemma 5.3 follows that  $x_\eta^\pm$  are linear maps from  $R$  and  $\Lambda$  to  $U_{\hbar} \mathfrak{h}$ . Moreover, we have  $[x_\eta^+(r), \tilde{f}[\epsilon]] = [[h^+[r], T[\eta]], \tilde{f}[\epsilon]] = -[[T[\eta], \tilde{f}[\epsilon]], h^+[r]] - [[\tilde{f}[\epsilon], h^+[r]], T[\eta]]$ ; both terms are zero by Prop. 5.3, so that we have

$$[x_\eta^+(r), \tilde{f}[\epsilon]] = 0;$$

in the same way, one shows that

$$[x_\eta^-(\lambda), \tilde{f}[\epsilon]] = 0.$$

But any element  $x$  of  $U_{\hbar} \mathfrak{h}$ , such that  $[x, \tilde{f}[\epsilon]] = 0$  for any  $\epsilon$ , is zero. To show this, one may divide  $x$  by the greatest possible power of  $\hbar$  and check that the same statement is true in the classical affine Kac-Moody algebra.  $\square$

### 5.3. Commutation of $T(z)$ with $f(w)$ .

**Lemma 5.4.**  *$T(z)$  may be written*

$$T(z) = \tilde{f}(z) \tilde{e}(z)_{z \rightarrow R} + \tilde{e}(z)_{z \rightarrow \Lambda} \tilde{f}(z) + \kappa(z),$$

where  $\kappa(z)$  belongs to  $U_{\hbar} \mathfrak{h}[[z, z^{-1}]]$ .

*Proof.* We have

$$\tilde{f}(z) \tilde{e}(z)_{z \rightarrow R} + \tilde{e}(z)_{z \rightarrow \Lambda} \tilde{f}(z) - T_0(z)$$

is equal to

$$[\tilde{e}(z)_{z \rightarrow \Lambda'}, \tilde{f}(z)_{z \rightarrow \Lambda'}] - [\tilde{e}(z)_{z \rightarrow R_{2\lambda}}, \tilde{f}(z)_{z \rightarrow R_{2\lambda}}]$$

and therefore belongs to  $U_{\hbar} \mathfrak{h}[[z, z^{-1}]]$ .  $\square$

We first show:

**Lemma 5.5.** *The commutator  $[T(z), \tilde{f}(w)]$  belongs to  $U_{\hbar}\mathfrak{h}U_{\hbar}\mathfrak{n}_+^{[1]}$ ; in other words, there are formal series  $K_i(z, w)$  in  $U_{\hbar}\mathfrak{h}[[z, z^{-1}, w, w^{-1}]]$ , such that*

$$[T(z), \tilde{f}(w)] = \sum_i K_i(z, w) \tilde{f}[\epsilon_i]. \quad (54)$$

*Proof.* It suffices to show this with  $T_0(z)$  instead of  $T(z)$ . This follows from a reasoning analogous to the first part of the proof of Prop. 5.1.  $\square$

From there follows:

**Proposition 5.3.**  *$T(z)$  commutes with  $\tilde{f}(w)$ .*

*Proof.* Let  $\epsilon$  belong to  $\mathcal{K}$ .  $\tilde{e}[\epsilon]$  commutes with the left side of (54), by Props. 5.3 and 5.1. Let us write that it commutes with the right side of this equality. We get  $\sum_i [\tilde{e}[\epsilon], K_i(z, w)] \tilde{f}[\epsilon_i] +$  element of  $U_{\hbar}\mathfrak{h} = 0$ . From there follows that  $[\tilde{e}[\epsilon], K_i(z, w)] = 0$ . The reasoning of the end of the proof of Prop. 5.2 applies to show that  $K_i(z, w)$  vanishes.  $\square$

Props. 5.1, 5.2 and 5.3 imply Thm. 5.1.  $\square$

*Remark 5. Classical limit.* Let us show that  $T(z)$  is, up to a scalar, a deformation of the Sugawara tensor. Let us denote by  $e_{cl}(z)$ ,  $h_{cl}(z)$  and  $f_{cl}(z)$  the generating currents of  $\mathfrak{g}$ . Then we have

$$\begin{aligned} e(z) &= e_{cl}(z) + O(\hbar), & f(z) &= f_{cl}(z) + O(\hbar), \\ k^+(z) &= 1 + \frac{\hbar}{2} h_{cl}(z)_{z \rightarrow \Lambda} + o(\hbar), & k^-(z) &= 1 + \frac{\hbar}{2} h_{cl}(z)_{z \rightarrow R} + o(\hbar), \\ k_R(z) &= 1 + O(\hbar^2), \text{ so that} \\ k_{tot}^+(z) &= [1 + \frac{\hbar}{2} q^{2\partial_z} (h_{cl}(z)_{z \rightarrow \Lambda}) + \hbar^2 s(z)] [1 + \frac{\hbar}{2} h_{cl}(z)_{z \rightarrow R} + \hbar^2 t(z)] + O(\hbar^3) \\ k_{tot}^-(z) &= [1 - \frac{\hbar}{2} q^{\partial_z} (h_{cl}(z)_{z \rightarrow \Lambda}) - \hbar^2 s(z) + \frac{\hbar^2}{4} (h_{cl}(z)_{z \rightarrow \Lambda})^2] \\ &\quad [1 - \frac{\hbar}{2} q^{-\partial_z} (h_{cl}(z)_{z \rightarrow \Lambda}) - \hbar^2 t(z) + \frac{\hbar^2}{4} (h_{cl}(z)_{z \rightarrow R})^2] + O(\hbar^3), \end{aligned}$$

where  $s(z)$  and  $t(z)$  are some currents. Then

$$\begin{aligned} T(z) &= e_{cl}(z)_{z \rightarrow \Lambda} f_{cl}(z) + f_{cl}(z) e_{cl}(z)_{z \rightarrow R} + \frac{1}{\hbar^2} (k_{tot}^+(z) + k_{tot}^-(z)) + O(\hbar) \\ &= \frac{1}{\hbar^2} + e_{cl}(z)_{z \rightarrow \Lambda} f_{cl}(z) + f_{cl}(z) e_{cl}(z)_{z \rightarrow R} + \frac{1}{2} \partial h_{cl}(z) \\ &\quad + \frac{1}{4} (h_{cl}(z)_{z \rightarrow \Lambda} h_{cl}(z) + h_{cl}(z) h_{cl}(z)_{z \rightarrow R}) + O(\hbar); \end{aligned}$$

so  $T(z) - \hbar^{-2}$  coincides with the classical Sugawara tensor to order  $\hbar$ .  $\square$

*Remark 6. Other expressions of  $T(z)$ .* One may show that up to an additive scalar constant,  $T(z)$  coincides with

$$\begin{aligned} T'(z) = & k^+(q^\partial z)^{-1} (f(z)_{z \rightarrow \Lambda'} e(q^\partial z) + e(q^\partial z) f(z)_{z \rightarrow R_{2\lambda}}) k^-(q^{-\partial} z) \\ & + \frac{\gamma'_\lambda(z)}{\hbar \sigma(z)} k^+(q^\partial z)^{-1} k^-(q^{-2\partial} z)^{-1} + \frac{\delta_{2\lambda}(z)}{\hbar \sigma(q^{2\partial} z)} k^+(q^{2\partial} z) k^-(q^{-\partial} z), \end{aligned} \quad (55)$$

with

$$\gamma'_\lambda(z) = \alpha(q^{2\partial} z) - \sigma(q^{2\partial} z) [g_\lambda^-(z) + \sum_i e^i(z) (q^\partial (q^{-2\partial} e_i)_R)(z)]$$

and

$$\delta_\lambda(z) = \beta(q^{2\partial} z) + \sigma(q^{2\partial} z) g_\lambda^+(z).$$

It also coincides with  $T''(z)$  defined by

$$\begin{aligned} T''(z) = & k^+(z)^{-1} (e(z)_{z \rightarrow \Lambda} f(q^{-\partial} z) + f(q^{-\partial} z) e(z)_{z \rightarrow R}) k^-(q^{-2\partial} z) \\ & + \frac{1}{\hbar} \frac{\alpha}{\sigma}(z) k^+(q^{-\partial} z) k^-(q^{-2\partial} z) + \frac{1}{\hbar} \frac{\beta'}{\sigma}(z) k^+(z)^{-1} k^-(q^{-\partial} z)^{-1}, \end{aligned} \quad (56)$$

up to an additive constant, with  $\beta'(z) = \beta(q^\partial z) - \sigma(q^\partial z) \sum_i q^\partial ((q^{-2\partial} e_i)_R)(z) e^i(z)$ .

This formula is a generalization of the formula given in [12], which uses [14] and the new realizations isomorphism. To see the correspondance between this formula and ours, let us modify the notation in [12] so that the quantum parameter of that paper is denoted by  $\underline{q}$ . The level in [12] is denoted by  $k$  and the currents generating the algebra  $U_q \mathfrak{g}$  are  $k_1^\pm(z)$ ,  $E(z)$  and  $F(z)$ .

Set  $X = \mathbb{C}P^1$  and  $\partial = z \frac{d}{dz}$ . The algebra  $U_{\hbar, \omega} \mathfrak{g}$  is isomorphic to  $U_q \mathfrak{g}$ , the isomorphism  $i$  being given by the formulas

$$i(K) = k, \quad i(k^+(z)) = k_1^+(z \underline{q}^{\frac{k}{2}+2})^{-1}, \quad i(k^-(z)) = k_1^-(z \underline{q}^{\frac{3k}{2}}),$$

$$i(e(z)) = -\frac{1}{\hbar(\underline{q} - \underline{q}^{-1})} E(z), \quad i(f(z)) = F(\underline{q}^k z),$$

with

$$q = \underline{q}^{-2}, \quad q(z, w) = \frac{q^{-1}z - w}{z - q^{-1}w}, \quad q_+(z, w) = \frac{q^{-1/2}z - q^{1/2}w}{z - w}.$$

Formula (6.10) of [12] then gives

$$\begin{aligned} i^{-1}(\ell(z)) = & \frac{1}{\hbar(\underline{q} - \underline{q}^{-1})} k^+(z)^{-1} : e(z) f(q^{-1}z) : k^-(z q^{-2}) \\ & + q^{-1/2} k^+(z q^{-1}) k^-(z q^{-2}) + q^{1/2} k^+(z)^{-1} k^-(z q^{-1})^{-1}, \end{aligned}$$

so  $i^{-1}(\ell(z))$  is equal to  $T(z)$  given by (56).  $\square$

*Remark 7. Genus 1 case.* Assume  $X$  is an elliptic curve  $\mathbb{C}/L$ ,  $L = \mathbb{Z} + \tau\mathbb{Z}$ , and  $\omega = dz$ . Let  $\theta$  be the Jacobi theta-function, equal to

$$\theta(z) = \frac{\sin(\pi z)}{\pi} \prod_{j=1}^{\infty} \frac{(1 - e^{2i\pi(j\tau+z)})(1 - e^{2i\pi(j\tau-z)})}{(1 - e^{2i\pi j\tau})^2}$$

The Weierstrass function is  $\wp = -(d/dz)^2 \ln \theta(z)$ . According to Prop. 1.1, we have  $R = \mathbb{C}1 \oplus (\oplus_{i \geq 0} \mathbb{C}(d/dz)^i \wp)$  and  $\Lambda = \mathbb{C}\theta'/\theta \oplus z\mathbb{C}[[z]]$ . We have also  $R_\lambda = \oplus_{i \geq 0} \mathbb{C}(d/dz)^i (\frac{\theta(z-\lambda)}{\theta(z)})$  and  $\Lambda' = \mathbb{C}[[z]]$ . We have

$$G(z, w) = d/dz \ln \theta(z - w) - d/dz \ln \theta(z) + d/dz \ln \theta(w),$$

$$G_{2\lambda}(z, w) = \frac{\theta(-2\lambda + z - w)}{\theta(z - w)\theta(-2\lambda)},$$

$$q_-(z, w) = \frac{\theta(z-w-\hbar)}{\theta(z-w)}, \text{ viewed as a series in } \mathbb{C}((z))((w))[[\hbar]],$$

$$\sigma(z) = \theta(\hbar), \quad \gamma'_\lambda(z) = \frac{\theta(2\lambda - \hbar)}{\theta(2\lambda)}, \quad \delta_\lambda(z) = \frac{\theta(2\lambda + \hbar)}{\theta(2\lambda)}.$$

The expression of  $T'(z)$  is then

$$\begin{aligned} T'(z) = & k^+(z + \hbar)^{-1} (f(z)_{z \rightarrow \Lambda'} e(z + \hbar) + e(z + \hbar) f(z)_{z \rightarrow R_{2\lambda}}) k^-(z - \hbar) \\ & + \frac{\theta(2\lambda - \hbar)}{\hbar \theta(\hbar) \theta(2\lambda)} k^+(z + \hbar)^{-1} k^-(z - 2\hbar)^{-1} + \frac{\theta(2\lambda + \hbar)}{\hbar \theta(2\lambda) \theta(\hbar)} k^+(z + 2\hbar) k^-(z - \hbar); \end{aligned}$$

we have also

$$T(z) = \tilde{e}(z) \tilde{f}(z)_{z \rightarrow R_{2\lambda}} + \tilde{f}(z)_{z \rightarrow \Lambda'} \tilde{e}(z) + \frac{1}{\hbar} \frac{\theta(2\lambda + \hbar)}{\theta(2\lambda) \theta(\hbar)} k_{tot}^+(z) + \frac{1}{\hbar} \frac{\theta(2\lambda - \hbar)}{\theta(2\lambda) \theta(\hbar)} k_{tot}^-(z).$$

□

## 6. SUBALGEBRAS $U_{\hbar} \mathfrak{g}^{out}$ AND $U_{\hbar} \mathfrak{g}_{\lambda_0}^{out}$ OF $U_{\hbar, \omega} \mathfrak{g}$ AND COPRODUCTS

**6.1. Subalgebras  $U_{\hbar} \mathfrak{g}^{out}$  and  $U_{\hbar} \mathfrak{g}_{\lambda_0}^{out}$ .** In [7], we showed that  $U_{\hbar, \omega} \mathfrak{g}$  contains a “regular” subalgebra  $U_{\hbar} \mathfrak{g}^{out}$ , generated by the  $h^+[r], e[r]$  and  $f[r]$ , for  $r$  in  $R$ . The inclusion  $U_{\hbar} \mathfrak{g}^{out} \subset U_{\hbar, \omega} \mathfrak{g}$  is a deformation of the inclusion of the classical enveloping algebra of  $\bar{\mathfrak{g}} \otimes R$  in that of  $\mathfrak{g} = (\bar{\mathfrak{g}} \otimes \mathcal{K}) \oplus \mathbb{C}K$ .

For any  $\lambda_0$  in  $\mathbb{C}^g$ , define  $U_{\hbar} \mathfrak{g}_{\lambda_0}^{out}$  as the subalgebra of  $U_{\hbar, \omega} \mathfrak{g}^{out}$ , generated by the  $h^+[r], e[r_{2\lambda_0}]$  and  $f[r_{-2\lambda_0}]$ , for  $r$  in  $R$ ,  $r_{\pm 2\lambda_0}$  in  $R_{\pm 2\lambda_0}$ .

**Proposition 6.1.** *Define  $\mathfrak{g}_{\lambda_0}^{out}$  to be the Lie algebra  $(\bar{\mathfrak{n}}_+ \otimes R_{2\lambda_0}) \oplus (\bar{\mathfrak{h}} \otimes R) \oplus (\bar{\mathfrak{n}}_- \otimes R_{-2\lambda_0})$ . The inclusion  $U_{\hbar} \mathfrak{g}_{\lambda_0}^{out} \subset U_{\hbar, \omega} \mathfrak{g}$  is a deformation of the inclusion of the classical enveloping algebra of  $\mathfrak{g}_{\lambda_0}^{out}$  in that of  $\mathfrak{g}$ .*

*Proof.* (12) implies that the  $e[\epsilon]$  satisfy the relations given by the pairing of

$$(1 + \psi(-\hbar, \partial_z^i \gamma) G(z, w)) e(z) e(w) = e^{2(\tau - \phi)} (1 + \psi(\hbar, \partial_z^i \gamma) G(z, w)) e(w) e(z)$$

with any  $v$  in  $\mathcal{K} \otimes \mathcal{K}$ , such that  $m(v) = 0$ , where  $m$  is the multiplication map.

Taking for  $v$  any  $\alpha \otimes \beta - \beta \otimes \alpha$ , with  $\alpha, \beta$  in  $R_{2\lambda_0}$ , and using the fact that  $R_{2\lambda_0}$  is an  $R$ -module, we get relations of the form

$$[e[\alpha], e[\beta]] = \sum_{i \geq 1, j} \hbar^i e[\alpha_j^{(i)}] e[\beta_j^{(i)}],$$

with  $\alpha_j^{(i)}, \beta_j^{(i)}$  in  $R_{2\lambda_0}$ . Therefore, if  $e_{2\lambda_0; i}$  is a basis of  $R_{2\lambda_0}$ , the family

$$(e[e_{2\lambda_0; i_1}] \cdots e[e_{2\lambda_0; i_p}])_{i_1 \leq \cdots \leq i_p}$$

spans the subalgebra of  $U_{\hbar, \omega} \mathfrak{g}$  generated by the  $e[r]$ ,  $r$  in  $R_{2\lambda_0}$ . Since by [7], Lemma 3.3, this is also a free family, it forms a basis of this subalgebra. To finish the proof, one proves the similar basis result for the subalgebra generated by the  $f[r]$ ,  $r$  in  $R_{-2\lambda_0}$  and a triangular decomposition result (see [7], Prop. 3.2 and Prop. 3.5).  $\square$

**6.2. Coproducts.** Set  $A = U_{\hbar, \omega} \mathfrak{g}$ ,  $B = U_{\hbar} \mathfrak{g}^{out}$ ,  $B_{\lambda_0} = U_{\hbar} \mathfrak{g}_{\lambda_0}^{out}$ .

Define for  $\mathbf{n} = (n_i)_{1 \leq i \leq p}$ ,  $I_{\mathbf{n}}$  as the left ideal of  $A$  generated by the  $x[\epsilon], \epsilon \in \prod_i z_i^{n_i} \mathbb{C}[[z_i]]$ . Define  $A \otimes_{>} A$ ,  $A \otimes_{<} A$  and  $A \bar{\otimes} A$  as the completions of  $A \otimes A$  with respect to the topologies defined by  $A \otimes I_{\mathbf{n}}$ ,  $I_{\mathbf{n}} \otimes A$  and  $I_{\mathbf{n}} \otimes A + A \otimes I_{\mathbf{n}}$  ( $\otimes$  denotes the  $\hbar$ -adically completed tensor product). We have the inclusions  $A \otimes_{>} A \subset A \bar{\otimes} A$ ,  $A \otimes_{<} A \subset A \bar{\otimes} A$  and  $A \otimes A = (A \otimes_{>} A) \cap (A \otimes_{<} A)$ .

We define also for any space  $V$ ,  $V \otimes_{>} A$  as the completion of  $V \otimes A$  w.r.t. the topology defined by the  $V \otimes I_{\mathbf{n}}$ ,  $A^{\otimes > n}$  as  $A^{\otimes > n-1} \otimes_{>} A$ , and  $A^{\otimes < n}$  in the same way.

In [7], we defined Drinfeld-type coproducts  $\Delta$  and  $\bar{\Delta}$  on  $U_{\hbar, \omega} \mathfrak{g}$  by formulas similar to those of [3].  $\Delta$  and  $\bar{\Delta}$  map  $A$  to  $A \otimes_{<} A$  and to  $A \otimes_{>} A$ . Moreover,  $\Delta$  and  $\bar{\Delta}$  are conjugated by an element  $F$  of  $A \bar{\otimes} A$ .  $F$  is decomposed as a product  $F_2 F_1$ , with  $F_1$  in  $A \otimes_{<} B$  and  $F_2$  in  $B \otimes_{>} A$ , which are defined as  $\lim_{\leftarrow} A \otimes B / I_{\mathbf{n}} \otimes B$  and  $\lim_{\leftarrow} B \otimes A / B \otimes I_{\mathbf{n}}$ .

$\Delta_R$  is defined as  $\text{Ad}(F_1) \circ \Delta$ . It maps therefore  $A$  to  $A \otimes_{<} A$ . Since  $\Delta_R$  is equal to  $\text{Ad}(F_2^{-1}) \circ \bar{\Delta}$ , it also maps  $A$  to  $A \otimes_{>} A$  and therefore to  $A \otimes A$ . Also we have  $\Delta_R(B) \subset B \otimes B$ .

**Theorem 6.1.** *We have  $\Delta(B_{\lambda_0}) \subset A \otimes_{<} B_{\lambda_0}$  and  $\bar{\Delta}(B_{\lambda_0}) \subset B_{\lambda_0} \otimes_{>} A$ . We have a decomposition*

$$F = F_{2; \lambda_0} F_{1; \lambda_0}, \text{ with } F_{1; \lambda_0} \in A \otimes_{<} B_{\lambda_0} \text{ and } F_{2; \lambda_0} \in B_{\lambda_0} \otimes_{>} A.$$

*Set  $\Delta_{\lambda_0} = \text{Ad}(F_{1; \lambda_0}) \circ \Delta$ , then  $\Delta_{\lambda_0}$  defines a quasi-Hopf algebra structure on  $A$ , for which  $B$  is a sub-quasi-Hopf algebra.*

*Sketch of proof.* The first statement is proved like Prop. 4.4 of [7], using the fact that  $R_{\pm 2\lambda_0}$  are  $R$ -modules. The decomposition of  $F$  is proved using the same duality arguments, e.g. the annihilator of  $U_{\hbar} \mathfrak{n}_+ \cap B_{2\lambda_0}$  in  $U_{\hbar} \mathfrak{n}_-$  is equal to  $\sum_{r \in R_{-2\lambda_0}} U_{\hbar} \mathfrak{n}_- f[r]$ . The proof of the next statements follows [7].  $\square$

7. FINITE DIMENSIONAL REPRESENTATIONS OF  $U_{h,\omega}\mathfrak{g}$ .

In [6], we constructed a family  $\pi_\zeta$  of 2-dimensional representations of  $U_{h,\omega}\mathfrak{g}$  at level zero, indexed by  $\zeta$  in the infinitesimal neighborhood  $\text{Spec}(\mathcal{K})$  of the  $P_i$ . We have

$$\pi_\zeta(K^+(z)) = \begin{pmatrix} q_-(z, \zeta) & 0 \\ 0 & q_-(q^\partial z, \zeta)^{-1} \end{pmatrix}, \quad \pi_\zeta(K^-(z)) = \begin{pmatrix} q_+(z, \zeta) & 0 \\ 0 & q_+(q^\partial z, \zeta)^{-1} \end{pmatrix},$$

$$\pi_\zeta(e(z)) = \begin{pmatrix} 0 & -\hbar\sigma(z)\delta(z, \zeta) \\ 0 & 0 \end{pmatrix}, \quad \pi_\zeta(f(z)) = \begin{pmatrix} 0 & 0 \\ \delta(z, \zeta) & 0 \end{pmatrix}.$$

This family extends to a family of representations of  $U_h\mathfrak{g}^{out}$ , indexed by  $\zeta$  in  $X - \{P_i\}$ . Formulas are

$$\pi_\zeta(K^+(z)) = \begin{pmatrix} q_-(z, \zeta) & 0 \\ 0 & q_-(q^\partial z, \zeta)^{-1} \end{pmatrix},$$

$$\pi_\zeta(e[r]) = \begin{pmatrix} 0 & -\hbar\sigma(\zeta)r(\zeta) \\ 0 & 0 \end{pmatrix}, \quad \pi_\zeta(f[r]) = \begin{pmatrix} 0 & 0 \\ r(\zeta) & 0 \end{pmatrix}.$$

It also extends to a family of representations of  $U_h\mathfrak{g}_{\lambda_0}^{out}$  by the same formulas, where we fix a preimage of  $\zeta$  in  $\tilde{X} - \pi^{-1}(P_0)$ . Changing the preimage of  $\zeta$  amounts to conjugating the representation by a diagonal matrix.

Define a parenthesis order on  $n$  objects as a binary tree with extremal vertices labelled  $1, \dots, n$ . To each such order, and to  $n$  points  $\zeta_i$  of  $X - \{P_i\}$ , we associate some  $B_{\lambda_0}$ -module. In the case of the representation  $V = ((V(\zeta_1) \otimes V(\zeta_2)) \otimes (V(\zeta_3) \otimes V(\zeta_4)))$ , the space of the representation is  $V = \otimes_{i=1}^n V(\zeta_i)$  and the morphism from  $B_{\lambda_0}$  to  $\text{End}(V)$  is  $(\otimes_{i=1}^n \pi_{\zeta_i}) \circ (\Delta \otimes \Delta) \circ \Delta$ .

In case the  $\zeta_i$  are formal and  $\zeta_1 << \zeta_2 << \dots << \zeta_n$ , the morphism  $\rho_V^{(P)}$  is the restriction of a morphism from  $A$  to  $\text{End}(V)$ , which is  $\otimes_{i=1}^n \pi_{\zeta_i} \circ \text{Ad}(\Delta^{(P)}(F_{1;\lambda_0}))$ . For example for  $V = ((V(\zeta_1) \otimes V(\zeta_2)) \otimes (V(\zeta_3) \otimes V(\zeta_4)))$ ,  $\Delta^{(P)}(F_1)$  is equal to  $\Delta^{(P)}(F_{1;\lambda_0}) = F_{1;\lambda_0}^{(12)} F_{1;\lambda_0}^{(34)} (\Delta \otimes \Delta)(F_{1;\lambda_0})$ .

Let  $(\varepsilon_1, \varepsilon_2)$  be the canonical basis of  $\mathbb{C}^2$ ,  $\xi_1, \xi_2$  its dual basis.

**Proposition 7.1.** *Let  $\zeta_1, \dots, \zeta_n$  be points of  $X - \{P_i\}$ ; let  $P$  be a parenthesis order, and define  $V$  as the  $B_{\lambda_0}$ -module  $\otimes_i^{(P)} V(\zeta_i)$  is then a  $B_{\lambda_0}$ -module; we denote by  $\rho_V^{(P)}$  the corresponding morphism from  $B_{\lambda_0}$  to  $\text{End}(V)$ . It has the following properties:*

- 1)  $e[r]$  and  $f[r]$  act on  $V$  as  $\sum_i A_i(\zeta_1, \dots, \zeta_n)r(\zeta_i)$ ,  $A_i$  in  $\text{End}(V) \otimes R^{\otimes n}$ ;
- 2) define the linear form  $\xi$  on  $V$  to be  $\otimes_{i=1}^n \xi_1^{(i)}$ . Then we have  $\langle \xi, \rho_V^{(P)}(f[r])v \rangle = 0$  for any  $r$  of  $R_{2\lambda_0}$  and  $\langle \xi, \rho_V^{(P)}(K^+(z))v \rangle = \prod_i q_+(z, \zeta_i) \langle \xi, v \rangle$ , for any  $v$  in  $V$ .

*Proof.* Let us first show 1) when  $\zeta_i$  are formal and  $\zeta_1 << \zeta_2 << \dots$ . In that case,  $\rho_V^{(P)}(e(z))$  is conjugate to  $\Delta^{(n)}(e(z))$ , which has the form

$$\sum_i A'_i \delta(z, \zeta_i) q_i(\zeta_i, \zeta_{i+1}, \dots, \zeta_n),$$

with  $A'_i$  some endomorphisms of  $V$  and  $q_i$  in  $\mathbb{C}((\zeta_i)) \cdots ((\zeta_n))$ . Therefore  $\Delta^{(n)}(e[r])$  is equal some  $\sum_i A'_i r(\zeta_i) q_i(\zeta_i, \dots, \zeta_n)$ . On the other hand,  $\rho_V^{(P)}(e[r])$  is equal to the conjugation of  $\Delta^{(n)}(e[r])$  by  $(\otimes_{i=1}^n \pi_{\zeta_i})(\Delta^{(P)}(F_{1;\lambda_0}))$ .  $\Delta^{(n)}(e[r])$  belongs to

$$\text{End}(V)((\zeta_1)) \cdots ((\zeta_n)),$$

so that  $\rho_V^{(P)}(e[r])$  has the form

$$\sum_i B_i(\zeta_1, \dots, \zeta_n) r(\zeta_i), \quad (57)$$

where  $B_i(\zeta_1, \dots, \zeta_n)$  belongs to  $\text{End}(V)((\zeta_1)) \cdots ((\zeta_n))$ .  $r$  being fixed,  $\rho_V^{(P)}(e[r])$  is an algebraic function in the  $\zeta_i$ , so the  $B_i(\zeta_1, \dots, \zeta_n)$  are algebraic functions and  $\rho_V^{(P)}(e[r])$  has the form (57) for any  $\zeta_i$  in  $X - \{P_i\}$ . This proves 1).

Let us prove 2). Since  $\Delta^{(P)} F_{1;\lambda_0}$  has total weight zero (i.e. it commutes with  $\Delta h[1] = \sum_i h[1]^{(i)}$ ),  $\rho_V^{(P)}(f[r])$  has weight  $-1$ , which implies the first statement. Let us prove the second statement. We can show by induction that

$$\langle \xi, \Delta^{(P)}(F_{1;\lambda_0}) \rangle = \langle \xi, v \rangle, \quad (58)$$

for any  $v$  in  $V$ . For example, in the case of a representation  $V = ((V(\zeta_1) \otimes V(\zeta_2)) \otimes (V(\zeta_3) \otimes V(\zeta_4)))$ , we have

$$\langle \xi, (\otimes_{i=1}^4 \pi_{\zeta_i})(F_{1;\lambda_0}^{(12)} F_{1;\lambda_0}^{(34)} (\Delta \otimes \Delta)(F_{1;\lambda_0}))(v) \rangle = \langle \xi, (\otimes_{i=1}^4 \pi_{\zeta_i})((\Delta \otimes \Delta)(F_{1;\lambda_0}))(v) \rangle$$

because  $F_{1;\lambda_0}$  belongs to  $1 + U_{\hbar \mathfrak{n}_+^{[\geq 1]}} \bar{\otimes} U_{\hbar \mathfrak{n}_-^{[\geq 1]}}$ ; as  $(\Delta \otimes \Delta)(F_{1;\lambda_0})$  belongs to

$$1 + (U_{\hbar \mathfrak{n}_+^{[\geq 1]}} \bar{\otimes} A^{\bar{\otimes} 3})[0] + (A \bar{\otimes} U_{\hbar \mathfrak{n}_+^{[\geq 1]}} \bar{\otimes} A^{\bar{\otimes} 2})[0]$$

(where  $[0]$  means the zero weight component w.r.t. the adjoint action of  $\sum_i h[1]^{(i)}$ ), we have

$$\langle \xi, (\Delta \otimes \Delta)(F_{1;\lambda_0})v \rangle = \langle \xi, v \rangle.$$

On the other hand, we have  $\langle \xi, (\otimes_{i=1}^4 \pi_{\zeta_i})(\Delta^{(n)}(K^+(z)))(v) \rangle = \prod_i q_+(z, \zeta_i) \langle \xi, v \rangle$ . Together with (58), this shows the statement for  $K^+(z)$ .

*Remark 8.* Prop. 7.1, 2) means that the “Drinfeld polynomial” of  $\otimes_i^{(P)} V(\zeta_i)$  is  $\prod_i q_+(q^\partial z, \zeta_i)$  (see [3]).  $\square$

Define  $k_{a \rightarrow R}(z)$  as  $\exp(\sum_{i,j} c_{ij} h[e^i] e^j(z))$ , where  $c_{ij}$  are as in (23).

**Corollary 7.1.** *There are formal series  $\pi_{\alpha, \zeta}(z)$  such that*

$$\langle \rho_V^{(P)}(k^+(q^{2\partial} z) k_R(q^\partial z) k_R(z)^{-1} k_{a \rightarrow R}(z) v), \xi \rangle = \prod_i \pi_{\alpha, \zeta_i}(z) \langle v, \xi \rangle,$$

for any  $v$  in  $V$ .

## 8. TWISTED CORRELATION FUNCTIONS

Let  $\mathbb{V}$  be a module over  $U_{\hbar,\omega}\mathfrak{g}/(K+2)$ . Let  $\psi_{\lambda_0}$  be a  $U_{\hbar}\mathfrak{g}_{\lambda_0}^{out}$ -module map from  $\mathbb{V}$  to  $V$  and set  $\psi_{\lambda} = \psi_{\lambda_0} \circ e^{\sum_a (\lambda_a - \lambda_a^{(0)})\hbar[r_a]}$ . Fix  $v$  in  $\mathbb{V}$  and let us set

$$f_{\lambda}(u_1, \dots, u_n) = \langle \psi_{\lambda}[\tilde{e}(u_1) \cdots \tilde{e}(u_n)v], \xi \rangle,$$

where  $\xi$  is the linear form defined in Prop. 7.1.

**Proposition 8.1.**  *$f_{\lambda}(u_1, \dots, u_n)$  is a symmetric function in  $(u_i)$ , such that*

$$[\prod_{i=1}^n \prod_j \pi_{\zeta_j}(q^{\partial} u_i)] f_{\lambda}(u_1, \dots, u_n)$$

*is regular on  $\tilde{X}^n$  except for poles for  $u_i$  at  $\pi^{-1}(P_j)$ , and simple poles for  $u_i$  at  $\pi^{-1}(q^{-\partial}\zeta_j)$ , and satisfy transformation properties (2), with  $\lambda_a^{(0)}$  replaced by  $\lambda_a$ .*

*When  $V$  is the trivial representation,  $f_{\lambda}(u_1, \dots, u_n)$  is regular on  $(\tilde{X} - \pi^{-1}(P_0))^n$ .*

*Proof.* From the commutation relations of  $\tilde{e}(z)$  follows that  $f_{\lambda}(u_1, \dots, u_n)$  is symmetric in the  $u_i$ . We have

$$\prod_j \pi_{\zeta_j}(q^{\partial} u_1) f_{\lambda}(u_1, \dots, u_n) = \langle \psi_{\lambda}[e(q^{\partial} u_1)w(u_1, \dots, u_n)], \xi \rangle.$$

The fact that  $\langle \psi_{\lambda}[e[r]w], \xi \rangle = 0$  for  $r$  in  $R_{2\lambda}$  vanishing at  $\zeta_i$  implies that

$$\prod_j \pi_{\zeta_j}(q^{\partial} u_1) f_{\lambda}(u_1, \dots, u_n)$$

belongs to  $[(\text{annihilator for } \langle, \rangle_{\mathcal{K}} \text{ of } \{r \in R_{2\lambda} | r(\zeta_i) = 0\}) \otimes \mathcal{K}^{n-1}][[\hbar]]$ . This annihilator is the space of functions on  $\tilde{X}$  with simple poles at  $\zeta_i$  and a pole at the  $P_i$ , satisfying (2).  $\square$

9. ACTION OF  $T(z)$  ON CORRELATION FUNCTIONS

Let us set

$$q_{\mathfrak{m}}(z, w) = (k_{\mathfrak{m}}(z), \tilde{e}(w)), \quad \kappa(z) = \frac{\exp(2\alpha(q^{-\partial}z, q^{-\partial}z))}{\exp(2\alpha(q^{\partial}z, q^{-\partial}z))} (k_a(q^{-\partial}z), k^-(q^{-\partial}z))^{-1}.$$

**Lemma 9.1.** *We have*

$$q_{\mathfrak{m}}(z, w) = \exp[2\hbar \sum_i (\frac{1}{1+q^{-\partial}} e^i)(z)((T+U)e_i)_{\mathfrak{m}}(w)], \quad (59)$$

$$\kappa(z) = \exp[2\alpha(z, z) + 2\alpha(q^{-\partial}z, q^{-\partial}z) - 2\alpha(q^{\partial}z, z) - 2\alpha(q^{-\partial}z, z)] \quad (60)$$

$$\exp[2\hbar \sum_i (\frac{1}{1+q^{-\partial}} e^i)(z)((q^{\partial}-1)(T+U)e_i)_{R(a)}(z)](k^-(z), k^-(q^{-\partial}z)).$$

$q_{\mathfrak{m}}(z, w)$  has the expansion

$$q_{\mathfrak{m}}(z, w) = i_{\mathfrak{m}}(z, w) \frac{q^{\partial} z - w}{z - w},$$

with  $i_{\mathfrak{m}}(z, w)$  in  $\mathbb{C}[[z, w]][z^{-1}, w^{-1}][[\hbar]]^{\times}$ .

*Proof.*  $(k_{\mathfrak{m}}(z), \tilde{e}(w))$  is equal to  $(k_a(z), \tilde{e}(w))^{-1}(k^{-}(z), \tilde{e}(w))$ . We have already seen that  $(k_a(z), \tilde{e}(w)) = (k_a(z), \tilde{f}(w))^{-1}$ . Then

$$\begin{aligned} (k_a(z), \tilde{e}(w)) &= (k_a(z), e(q^{\partial} w)) = (k_a(z), f(q^{\partial} w))^{-1} \\ &= (k_a(z), \tilde{f}(q^{\partial} w) k^{-}(w)^{-1})^{-1} = (k_a(z), \tilde{f}(w))^{-1}. \end{aligned}$$

Therefore  $q_{\mathfrak{m}}(z, w) = (k_{\mathfrak{m}}(z), \tilde{f}(w))^{-1}$  and by (27), we get the statement on  $q_{\mathfrak{m}}(z, w)$ .  $\square$

Fix  $\Pi$  in  $\mathcal{K}[[\hbar]]$ . Let  $U$  be an open subset of  $\mathbb{C}^g$  and define  $\mathcal{F}_U$  as the space of functions  $f(\lambda_a | u_1, \dots, u_n)$  on  $U \times (\tilde{X} - \pi^{-1}(P_0))^n$ , symmetric in  $(u_1, \dots, u_n)$  and with transformation properties (2), with  $(\lambda_a^{(0)})$  replaced by  $\lambda_a$ . For  $f$  in  $\mathcal{F}_U$ , set

$$(T_z^{(\Pi)} f)(\lambda_a | u_1, \dots, u_n) \quad (61)$$

$$\begin{aligned} &= \Pi(z) a_{\lambda}(z) \prod_{i=1}^n q_{\mathfrak{m}}(z, u_i) f(\lambda_a + \hbar(\frac{1}{1+q^{-\partial}} \omega_a / \omega)(z) | u_1, \dots, u_n) \\ &+ \Pi(q^{-\partial} z)^{-1} b'_{\lambda}(z) \kappa(z) \prod_{i=1}^n q_{\mathfrak{m}}(q^{-\partial} z, u_i)^{-1} f(\lambda_a - \hbar(\frac{1}{1+q^{\partial}} \omega_a / \omega)(z) | u_1, \dots, u_n) \\ &+ \sum_i -\frac{1}{\hbar} \Pi(u_i) G_{2\lambda}(z, q^{\partial} u_i) q_{\mathfrak{m}}(u_i, z) \prod_{j \neq i} q_{\mathfrak{m}}(u_i, u_j) \\ &f(\lambda_a + \hbar(\frac{1}{1+q^{-\partial}} \omega_a / \omega)(u_j) | u_1, \dots, z, \dots, u_n) \\ &+ \sum_i \frac{1}{\hbar} \Pi(q^{-\partial} u_i)^{-1} G_{2\lambda}(z, q^{-\partial} u_i) \kappa(u_i) q_{\mathfrak{m}}(q^{-\partial} u_i, z)^{-1} \prod_{j \neq i} q_{\mathfrak{m}}(q^{-\partial} u_i, u_j)^{-1} \cdot \\ &\cdot f(\lambda_a - \hbar(\frac{1}{1+q^{\partial}} \omega_a / \omega)(u_i) | u_1, \dots, z, \dots, u_n) \end{aligned}$$

where  $b'_{\lambda}(z)$  is defined by (50) and in the two last sums,  $q_{\mathfrak{m}}(u_i, z)$ ,  $q_{\mathfrak{m}}(q^{-\partial} u_i, z)$ ,  $q_{\mathfrak{m}}(u_i, u_j)$  and  $q_{\mathfrak{m}}(q^{-\partial} u_i, u_j)$ ,  $j < i$  are continued to the domains  $u_i \ll z$  and  $u_i \ll u_j$ .

**Proposition 9.1.** *Assume that  $K$  acts by  $-2$  on  $\mathbb{V}$  and  $v$  is such that  $h[1]v = -2nv$ ,  $\tilde{h}[\epsilon]v = 0$  for  $\epsilon$  in  $\mathfrak{m}$ , and  $\tilde{f}[z^{1-g+k}]v = 0$  for  $k \geq 0$ . We have*

$$\langle \psi_{\lambda}[T(z)\tilde{e}(u_1) \cdots \tilde{e}(u_n)v], \xi \rangle = T_z^{(\Pi)}(\langle \psi_{\lambda}[\tilde{e}(u_1) \cdots \tilde{e}(u_n)v], \xi \rangle),$$

with  $T_z^{(\Pi)}$  defined by (61), and  $\Pi(z) = \prod_i \pi_{\alpha, \zeta_i}(z)$ .

*Proof.* We have

$$\begin{aligned} & \langle \psi_\lambda [T(z) \tilde{e}(u_1) \cdots \tilde{e}(u_n) v], \xi \rangle \\ &= \sum_i \langle \psi_\lambda [\tilde{e}(z) \tilde{e}(u_1) \cdots [\tilde{f}(z), \tilde{e}(u_i)]_{z \rightarrow R_{2\lambda}} \cdots \tilde{e}(u_n) v], \xi \rangle \\ &+ a_\lambda(z) \langle \psi_\lambda [k_{tot}^+(z) \tilde{e}(u_1) \cdots \tilde{e}(u_n) v], \xi \rangle + b_\lambda(z) \langle \psi_\lambda [k_{tot}^-(z) \tilde{e}(u_1) \cdots \tilde{e}(u_n) v], \xi \rangle, \end{aligned}$$

by the invariance of  $\psi_\lambda$ .

The sum is equal to

$$\begin{aligned} & \sum_i -\frac{1}{\hbar} G_{2\lambda}(z, q^\partial u_i) \langle \psi_\lambda [\tilde{e}(z) \tilde{e}(u_1) \cdots k_{tot}^+(u_i) \cdots \tilde{e}(u_n) v], \xi \rangle \\ &+ \frac{1}{\hbar} G_{2\lambda}(z, q^{-\partial} u_i) \frac{\exp(2\alpha(q^{-\partial} u_i, q^{-\partial} u_i))}{\exp(2\alpha(q^\partial u_i, q^{-\partial} u_i))} \langle \psi_\lambda [\tilde{e}(z) \tilde{e}(u_1) \cdots k_{tot}^-(u_i) \cdots \tilde{e}(u_n) v], \xi \rangle. \end{aligned}$$

Then

$$\begin{aligned} & \langle \psi_\lambda [k_{tot}^+(z) \tilde{e}(u_1) \cdots \tilde{e}(u_n) v], \xi \rangle \\ &= \Pi(z) \langle \psi_{\lambda_a + \sum_i b_{ai} e^i(z)} [k_m(z) \tilde{e}(u_1) \cdots \tilde{e}(u_n) v], \xi \rangle \\ &= \Pi(z) \prod_{i=1}^n (k_m(z), \tilde{e}(u_i)) \langle \psi_{\lambda_a + \sum_i b_{ai} e^i(z)} [\tilde{e}(u_1) \cdots \tilde{e}(u_n) v], \xi \rangle \\ &= \Pi(z) \prod_{i=1}^n q_m(z, u_i) \langle \psi_{\lambda_a + \sum_i b_{ai} e^i(z)} [\tilde{e}(u_1) \cdots \tilde{e}(u_n) v], \xi \rangle; \end{aligned}$$

the second equality follows from the covariance of  $\psi_\lambda$ , the next follows from the fact that  $v$  is  $U_{\hbar} \mathfrak{b}^{\geq 1-g}$ -invariant; in the same way

$$\begin{aligned} & \langle \psi_\lambda [k_{tot}^-(z) \tilde{e}(u_1) \cdots \tilde{e}(u_n) v], \xi \rangle \\ &= \Pi(q^{-\partial} z)^{-1} (k_a(q^{-\partial} z), k_m(q^{-\partial} z))^{-1} \langle \psi_{\lambda_a - \sum_i b_{ai} e^i(q^{-\partial} z)} [k_m(q^{-\partial} z)^{-1} \tilde{e}(u_1) \cdots \tilde{e}(u_n) v], \xi \rangle \\ &= \Pi(q^{-\partial} z)^{-1} (k_a(q^{-\partial} z), k^-(q^{-\partial} z))^{-1} \prod_{i=1}^n (k_m(q^{-\partial} z), \tilde{e}(u_i))^{-1} \\ & \langle \psi_{\lambda_a - \sum_i b_{ai} e^i(q^{-\partial} z)} [\tilde{e}(u_1) \cdots \tilde{e}(u_n) v], \xi \rangle \\ &= \Pi(q^{-\partial} z)^{-1} (k_a(q^{-\partial} z), k^-(q^{-\partial} z))^{-1} \prod_{i=1}^n q_m(q^{-\partial} z, u_i)^{-1} \\ & \langle \psi_{\lambda_a - \sum_i b_{ai} e^i(q^{-\partial} z)} [\tilde{e}(u_1) \cdots \tilde{e}(u_n) v], \xi \rangle, \end{aligned}$$

We have

$$-2(1 \otimes q^\partial \mathcal{A}) \beta(z, w) \in (R \otimes R)[[\hbar]] - 2\hbar \sum_a \left( \frac{1}{1 + q^{-\partial}} \omega_a / \omega \right)(z) r_a(w)$$

so

$$\beta(z, w) \in \hbar \sum_a \left( \frac{1}{1 + q^{-\partial}} \omega_a / \omega \right)(z) r_a(w) + (R \otimes R)[[\hbar]],$$

therefore

$$\sum_i b_{ai} e^i(z) = \hbar \left( \frac{1}{1 + q^{-\partial}} (\omega_a / \omega) \right)(z)$$

and the Proposition follows.  $\square$

*Remark 9. Dependence on  $\alpha$ .* The operators  $T_z$  depend on the choice of  $\alpha$  through their coefficients  $\kappa(z)$  and  $q_m(z, w)$ . Operators  $T_z$  corresponding to different choices  $\alpha$  and  $\alpha'$  are conjugated. When  $\Pi(z) = 1$ , the conjugation is  $T_z^{(\alpha)} = M_{\alpha\alpha'} T_z^{(\alpha')} M_{\alpha\alpha'}^{-1}$ , where

$$(M_{\alpha\alpha'} f)(\lambda_a | u_1, \dots, u_n) = \prod_{i < j} \exp[2(\alpha - \alpha')(q^\partial u_i, u_j)] f(\lambda_a | u_1, \dots, u_n).$$

## 10. COMMUTING DIFFERENCE OPERATORS

Define  $U_{\mathfrak{g}_{in}}^{\geq 1-g}$  as the subalgebra of  $U_{\hbar, \omega} \mathfrak{g}$  generated by  $h[1]$ , the  $\tilde{h}[\epsilon], \epsilon \in \mathfrak{m}$ , and the  $\tilde{f}[z^{1-g+k}], k \geq 0$ .

Let  $\chi_n$  be the character of  $U_{\mathfrak{g}_{in}}^{\geq 1-g}$  defined by  $\chi_n(h[1]) = -2n$ ,  $\chi_n(\tilde{h}[\epsilon]) = \chi_n(\tilde{f}[z^{1-g+k}]) = 0$ ,  $\epsilon \in \mathfrak{m}$ ,  $k \geq 0$ .

Define  $\mathbb{V}_n$  as the  $U_{\hbar, \omega} \mathfrak{g}$ -module  $U_{\hbar, \omega} \mathfrak{g} \otimes_{U_{\mathfrak{g}_{in}}^{\geq 1-g}} \mathbb{C}_{\chi_n}$ .

**Proposition 10.1.** *The map  $\iota$  from  $(\mathbb{V}_n^*)^{U_{\hbar} \mathfrak{g}_{\lambda_0}^{out}}$  to the subspace  $\mathcal{F}$  of  $S^n \mathcal{K}[[\lambda_a - \lambda_a^{(0)}]][[\hbar]]$  formed of the formal functions near  $\lambda_0$ , which can be continued in variables  $u_i$  to functions on  $\tilde{X} - \pi^{-1}(\{P_i\})$  with transformation properties (2) (with  $\lambda_a^{(0)}$  replaced by  $\lambda_a$ ), defined by*

$$\psi_{\lambda_0} \mapsto \langle \psi_{\lambda}, \tilde{e}(u_1) \cdots \tilde{e}(u_n) v \rangle,$$

is an isomorphism.

*Proof.*  $(\mathbb{V}_n^*)^{U_{\hbar} \mathfrak{g}_{\lambda_0}^{out}}$  is isomorphic to the space of forms  $\phi$  on  $U_{\hbar, \omega} \mathfrak{g}$  such that  $\phi(x^{out} x) = \varepsilon(x^{out}) \phi(x)$ ,  $x^{out}$  in  $U_{\hbar} \mathfrak{g}_{\lambda_0}^{out}$  and  $\phi(x x^{in}) = \phi(x) \chi_n(x^{in})$ ,  $x^{in}$  in  $U_{\mathfrak{g}_{in}}^{\geq 1-g}$ .

From Prop. 3.2 follows that the kernel of the product map

$$\tilde{\pi} : U_{\hbar} \mathfrak{g}_{\lambda_0}^{out} \otimes \mathbb{C} \langle h[r_a], \tilde{e}[\epsilon], \epsilon \in \mathcal{K} \rangle \otimes U_{\mathfrak{g}_{in}}^{\geq 1-g} \rightarrow U_{\hbar, \omega} \mathfrak{g} \quad (62)$$

is spanned by the  $x \tilde{e}[r_{-2\lambda_0}] \otimes y \otimes z - x \otimes \tilde{e}[r_{-2\lambda_0}] y \otimes z$ ,  $r_{-2\lambda_0}$  in  $R_{-2\lambda_0}$ ,  $x$  in  $U_{\hbar} \mathfrak{g}_{\lambda_0}^{out}$ ,  $y$  in  $\mathbb{C} \langle h[r_a], \tilde{e}[\epsilon], \epsilon \in \mathcal{K} \rangle$ ,  $z$  in  $U_{\mathfrak{g}_{in}}^{\geq 1-g}$ . An element of  $\mathcal{F}$  induces a form  $\bar{\phi}$  on  $\mathbb{C} \langle h[r_a], \tilde{e}[\epsilon], \epsilon \in \mathcal{K} \rangle$ , that we extend to the left side of (62) by the rule  $\bar{\phi}(x \otimes y \otimes z) = \varepsilon(x) \phi(y) \chi_n(z)$ . The properties of the elements of  $\mathcal{F}$  imply that  $\phi$  maps  $\text{Ker } \tilde{\pi}$  to zero. It follows that  $\iota$  is surjective.

In the same way, if  $\iota(\psi_{\lambda_0}) = 0$ , then the restriction of  $\iota(\psi_{\lambda_0})$  to  $\mathbb{C} \langle h[r_a], \tilde{e}[\epsilon], \epsilon \in \mathcal{K} \rangle$  is zero, so that  $\psi_{\lambda_0}$  is zero.  $\square$

**Theorem 10.1.** *For any  $\Pi(z)$  in  $\mathcal{K}[[\hbar]]$  and  $z$  in  $\text{Spec}(\mathcal{K})$ , the operators  $T_z^{(\Pi)}$  defined by (61) form a commuting family of evaluation-difference operators, acting on  $S^n(\mathcal{K})[[\lambda_a - \lambda_a^{(0)}]][[\hbar]]$ .*

*When  $\Pi(z) = 1$ , they form a commuting family of endomorphisms of  $\mathcal{F}_U[[\hbar]]$ , where  $\mathcal{F}_U$  is defined in sect. 9, for  $z$  in  $X - \{P_0\}$ . Set for  $\rho = (\rho_\lambda)_{\lambda \in \text{Spec } \mathbb{C}[[\lambda_a - \lambda_a^{(0)}]]}$  a family of elements of  $R_{-2\lambda} \cap z^{-N}\mathcal{O}$ ,*

$$\begin{aligned} & (\hat{f}[\rho]f)(\lambda_a|u_1, \dots, u_{n+1}) \\ &= \sum_{i=1}^{n+1} \frac{1}{\hbar} \rho_\lambda(u_i) \Pi(u_i) \prod_{j \neq i} q_m(u_i, u_j) f(\lambda_a + \frac{\hbar}{1+q^{-\partial}} \omega_a/\omega(u_i)|u_1, \dots, \check{i} \dots u_{n+1}) \\ & - \sum_{i=1}^{n+1} \frac{1}{\hbar} \rho_\lambda(q^{-\partial} u_i) \Pi(q^{-\partial} u_i)^{-1} \kappa(u_i) \prod_{j \neq i} q_m(q^{-\partial} u_i, u_j)^{-1} f(\lambda_a - \frac{\hbar}{1+q^{\partial}} \omega_a/\omega(u_i)|u_1, \dots, \check{i} \dots u_{n+1}). \end{aligned} \tag{63}$$

*Then the operators  $T_z^{(\Pi=1)}$  normalize the  $\hat{f}[\rho]$ , which means that they preserve the intersection  $\cap_{\rho_\lambda \in R_{-2\lambda} \cap z^{-N}\mathcal{O}} \text{Ker } \hat{f}[\rho]$  for any integer  $N$ .*

*Proof.* When  $\Pi = 1$ , the operators  $T_z^{(\Pi)}$  can be identified with the action of  $T(z)$  on the space of invariant forms  $(\mathbb{V}_n^*)^{U_{\hbar} \mathfrak{g}_{\lambda_0}^{\text{out}}}$ , by Prop. 10.1. Therefore, they preserve this space and commute with each other.

It follows that we have the cancellations of poles

$$\text{res}_{z=w} [a_\lambda(z) q_m(z, w) dz] = \text{res}_{z=w} \left[ \frac{1}{\hbar} G_{2\lambda}(z, q^\partial w) q_m(w, z) dz \right],$$

and

$$\text{res}_{z=w} [b'_\lambda(z) \kappa(z) q_m(q^{-\partial} z, w)^{-1} dz] + \text{res}_{z=w} \left[ \frac{1}{\hbar} G_{2\lambda}(z, q^{-\partial} w) q_m(q^{-\partial} w, z)^{-1} dz \right] = 0.$$

These relations imply that when  $\Pi$  is arbitrary,  $T_z^{(\Pi)}$  is a well-defined endomorphism of  $S^n(\mathcal{K})[[\hbar]]$ .

Set then  $\Pi^+(z) = \Pi(z)$ ,  $\Pi^-(z) = \Pi(q^{-\partial} z)^{-1}$ , and

$$[m(\varphi(u_i))f](\lambda_a|u_1, \dots, u_n) = \varphi(u_i) f(\lambda_a|u_1, \dots, u_n),$$

for any  $\varphi$  in  $\mathcal{K}[[\hbar]]$ , and

$$T_z^{(\Pi)} = \sum_{\epsilon=+,-} \Pi^\epsilon(z) A_z^\epsilon + \sum_{\epsilon=+,-} \sum_{i=1}^n m(\Pi^\epsilon(u_i)) \circ C_z^{\epsilon,(i)}.$$

Comparison of arguments in  $(\lambda_a)$  in the relation  $[T_z^{(\Pi=1)}, T_w^{(\Pi=1)}] = 0$  yields

$$[A_z^\epsilon, A_w^{\epsilon'}] = 0, \quad [C_z^{\epsilon,(i)}, C_w^{\epsilon',(j)}] = 0$$

for any  $\epsilon, \epsilon'$  and if  $i \neq j$  and

$$[A_z^\epsilon, C_w^{\epsilon',(j)}] + C_z^{\epsilon',(j)} C_w^{\epsilon,(j)} = 0.$$

On the other hand, we have

$$[m(\Pi^\epsilon(u_i)) \circ C_z^{\epsilon, (i)}, m(\Pi^{\epsilon'}(u_j)) \circ C_w^{\epsilon', (j)}] = m(\Pi^\epsilon(u_i)) \circ m(\Pi^{\epsilon'}(u_j)) \circ [C_z^{\epsilon, (i)}, C_w^{\epsilon', (j)}] = 0$$

for any  $\epsilon, \epsilon', i \neq j$ , and

$$\begin{aligned} & [\Pi^\epsilon(z)A_z^\epsilon, m(\Pi^{\epsilon'}(u_j)) \circ C_w^{\epsilon', (j)}] + m(\Pi^{\epsilon'}(u_j)) \circ C_z^{\epsilon', (j)} \circ m(\Pi^\epsilon(u_j)) \circ C_w^{\epsilon, (j)} \\ &= \pi^\epsilon(z)m(\Pi^{\epsilon'}(u_j)) \circ ([A_z^\epsilon, C_w^{\epsilon', (j)}] + C_z^{\epsilon', (j)}C_w^{\epsilon, (j)}) \\ &= 0. \end{aligned}$$

Therefore  $[T_z^{(\Pi)}, T_w^{(\Pi)}] = 0$ .

The statement on  $\hat{f}[\rho]$  follows from the fact that  $\cap_{\rho_\lambda \in R_{-2\lambda} \cap z^{-N} \mathcal{O}} \text{Ker } \hat{f}[\rho]$  is equal to  $(\mathbb{V}_{n,N}^*)^{U_{\hbar} \mathfrak{g}_{\lambda_0}^{out}}$ , where  $\mathbb{V}_{n,N}$  is the  $U_{\hbar, \omega} \mathfrak{g}$ -module  $U_{\hbar, \omega} \mathfrak{g} \otimes_{U_{\mathfrak{g}_{in}}^{\geq -N}} \mathbb{C}_{\chi_n}$ ,  $U_{\mathfrak{g}_{in}}^{\geq -N}$  is the subalgebra of  $U_{\hbar, \omega} \mathfrak{g}$  generated by the  $\tilde{h}[\epsilon], \epsilon \in \mathfrak{m}$ ,  $h[1]$  and the  $\tilde{f}[z^k], k \geq -N$ , and  $\chi_n$  is the character of this algebra defined by  $\chi_n(h[1]) = -2n$ ,  $\chi_n(\tilde{h}[\epsilon]) = \chi_n(\tilde{f}[z^k]) = 0$  for  $k \geq -N$  and  $\epsilon$  in  $\mathfrak{m}$ .  $\square$

*Remark 10.* Write  $k^+(q^{2\partial}z)k_R(q^\partial z)k_R(z)^{-1}k_{a \rightarrow R}(z) = \exp(\sum_i h[e^i]\rho_i(z))$ , with  $\rho_i(z)$  in  $\mathcal{K}[[\hbar]]$ . If  $\Pi(z)$  has the form  $\exp(\sum_i \lambda_i \rho_i(z))$ , for some  $\lambda_i$  in  $\mathbb{C}[[\hbar]]$ , then  $T_z^{(\Pi)}$  may be interpreted as the action of  $T(z)$  on some space of intertwiners.

## 11. CONNECTION WITH HYPERGEOMETRIC SPACES

In [16], V. Tarasov and A. Varchenko proved the following result. Let  $W$  be a representation of the Yangian  $Y(\mathfrak{sl}_2)$  and let  $\xi$  be a vector of  $W$  such that  $l_{21}^+(z)\xi = 0$ , and  $l_{ii}^+(z)\xi = \pi_i(z)\xi$ ,  $i = 1, 2$ , for  $\pi_i(z)$  some formal series.

**Proposition 11.1.** (see [16]). *We can express  $(l_{11}^+(z) + l_{22}^+(z))l_{12}^+(u_1) \cdots l_{12}^+(u_n)\xi$  in the form*

$$A(z|u_1, \dots, u_n)l_{12}^+(u_1) \cdots l_{12}^+(u_n)\xi + \sum_{i=1}^n C^{(i)}(z|u_1, \dots, u_n)l_{12}^+(u_1) \cdots l_{12}^+(z) \cdots l_{12}^+(u_n)\xi;$$

*the family of operators acting on symmetric functions of  $(u_1, \dots, u_n)$  defined by*

$$\hat{T}_z = A(z|u_1, \dots, u_n) + \sum_{i=1}^n C^{(i)}(z|u_1, \dots, u_n) \circ \text{ev}_z^{(i)}$$

*is commutative.*

In this section, we will show that the operators  $\hat{T}_z$  are examples of the operators  $T_z^{(\Pi)}$  constructed above.

Let us consider now the case  $X = \mathbb{CP}^1$ ,  $\omega = dz$ . We have  $\sum_i n_i P_i = 2(\infty)$ .  $U_{\hbar, \omega} \mathfrak{g}$  is then a completion of the central extension  $\widehat{DY}(\mathfrak{sl}_2)$  of the double of the Yangian  $Y(\mathfrak{sl}_2)$  of  $\mathfrak{sl}_2$ . Let  $x[t^n]$ ,  $x \in \{e, f, h\}$ ,  $n \in \mathbb{Z}$  be the “new realizations”

generators of  $DY(\mathfrak{sl}_2)$  and  $l_{ij}[n]$ ,  $1 \leq i, j \leq 2$  and  $n \in \mathbb{Z}$  its “matrix elements” generators.

Generators  $x[t^n]$  are organized in generating series  $e(z)$ ,  $f(z)$  and  $k^\pm(z)$ , as above; we further split  $x(z)$  as the sum  $x^+(z) + x^-(z)$ , with  $x^+(z) = \sum_{n \geq 0} x[t^n]z^{-n-1}$ ,  $x^-(z) = \sum_{n < 0} x[t^n]z^{-n-1}$ . Generating series for the  $l_{ij}[n]$  are  $l_{ij}^+(z) = \sum_{n \geq 0} l_{ij}[n]z^{-n-1}$ ,  $l_{ij}^-(z) = \sum_{n < 0} l_{ij}[n]z^{-n-1}$ .

We have the relations

$$(z - w + \hbar)e(z)e(w) = (z - w - \hbar)e(w)e(z),$$

$$k^+(z)e(w)k^+(z)^{-1} = \frac{z - w + \hbar}{z - w}e(w), \quad k^-(z)e(w)k^-(z)^{-1} = \frac{w - z + \hbar K}{w - z + \hbar(K + 1)}e(w),$$

and

$$l_{12}^+(z) = -\hbar k^+(z)^{-1}e^+(z), \quad l_{12}^-(z) = -\hbar e^-(z - \hbar K)k^-(z - \hbar)$$

(see e.g. [5]). Moreover, we have

$$\tilde{e}(z) = k^+(z + \hbar)^{-1}e(z + \hbar). \quad (64)$$

Define  $Y^{\geq 0}$  and  $Y^{< 0}$  as the subalgebras of  $\widehat{DY}(\mathfrak{sl}_2)$  generated the  $x[t^n]$ ,  $n \geq 0$  (resp. by the  $x[t^n]$ ,  $n < 0$ ). Let  $\widehat{Y}^{< 0}$  be the subalgebra generated by  $K$  and  $Y^{< 0}$ . Then  $U_{\hbar}\mathfrak{g}^{out}$  is equal to  $Y^{\geq 0}$ . Define  $\mathbb{V}$  as the Weyl module  $\widehat{DY}(\mathfrak{sl}_2) \otimes_{\widehat{Y}^{< 0}} \mathbb{C}_{-2}$ , where  $\mathbb{C}_{-2}$  is one-dimensional module over  $\widehat{Y}^{< 0}$  where all the generators act by zero, except for  $K$ , which acts by  $-2$ .

Let  $\zeta_i$  be points of  $\mathbb{C}$  and  $V_i(\zeta_i)$  be evaluation modules over  $Y^{\geq 0}$  associated with these points;  $V_i$  is  $(2\Lambda_i + 1)$ -dimensional. Define  $V$  as the tensor product (for the usual comultiplication of  $Y^{\geq 0}$ ) of the  $V_i(\zeta_i)$ . Let  $\psi$  be some  $Y^{\geq 0}$ -module map from  $\mathbb{V}$  to  $V$ . We will view  $V^*$  as a  $Y^{\geq 0}$ -module by the rule

$$\langle a\alpha, v \rangle = \langle \alpha, S(a)v \rangle$$

for  $a$  in  $Y^{\geq 0}$ ,  $v$  in  $V$  and  $\alpha$  in  $V^*$ , where  $S$  is the antipode of  $Y^{\geq 0}$ .

Let  $\xi$  be a highest weight linear form as in Prop. 7.1 and let  $\Omega$  be any vector of  $\mathbb{V}$  annihilated by the  $e^-(z)$  (for example,  $\Omega$  could be the vector  $1 \otimes 1$  of  $\mathbb{V}$ ). We have

$$\langle \xi, k^+(z)v \rangle = \pi_V(z)\langle \xi, v \rangle,$$

with

$$\pi_V(z)\pi_V(z + \hbar) = \prod_i \frac{\zeta_i - z + \hbar(2\Lambda_i + 1)}{\zeta_i - z}$$

(see [2]).

**Lemma 11.1.** *Let  $\tilde{\xi}$  be any linear form of  $\mathbb{V}$  such that*

$$\langle \tilde{\xi}, k^+(z)v \rangle = \pi(z)\langle \tilde{\xi}, v \rangle, \quad (65)$$

for any  $v$  in  $\mathbb{V}$  and some  $\pi(z)$  in  $\mathbb{C}[[z^{-1}]]$ . Then we have

$$\langle \tilde{\xi}, e(z_1) \cdots e(z_n) \Omega \rangle = \frac{1}{(-\hbar)^n} \prod_{i < j} \frac{z_j - z_i}{z_j - z_i - \hbar} \pi(z_1) \cdots \pi(z_n) \langle \tilde{\xi}, l_{12}^+(z_1) \cdots l_{12}^+(z_n) \Omega \rangle \quad (66)$$

(identity in  $\mathbb{C}((z_1)) \cdots ((z_n))$ ). In particular, we have

$$\begin{aligned} & \langle \psi(e(z_1) \cdots e(z_n) \Omega), \xi \rangle \\ &= \frac{1}{\hbar^n} \prod_{i < j} \frac{z_j - z_i}{z_j - z_i - \hbar} \pi_V(z_1) \cdots \pi_V(z_n) \langle \psi(\Omega), l_{12}^+(z_1 - \hbar) \cdots l_{12}^+(z_n - \hbar) \xi \rangle. \end{aligned} \quad (67)$$

*Proof.* We proceed by induction. For  $n = 0$ , the statement is trivial. Assume we have proved it at step  $n$  and let us try to prove it at step  $n + 1$ . Apply the statement of step  $n$  for  $\tilde{\xi}' = \tilde{\xi} \circ e(z_0)$ .  $\tilde{\xi}'$  satisfies (65) with  $\pi(z)$  replaced by  $\pi(z) \frac{z - z_0}{z - z_0 - \hbar}$ . Therefore, we have

$$\begin{aligned} \langle \tilde{\xi}, e(z_0) \cdots e(z_n) \Omega \rangle &= \langle \tilde{\xi}', e(z_1) \cdots e(z_n) \Omega \rangle \\ &= \frac{1}{(-\hbar)^n} \pi(z_1) \cdots \pi(z_n) \prod_{0 \leq i < j \leq n} \frac{z_j - z_i}{z_j - z_i - \hbar} \langle \tilde{\xi}', l_{12}^+(z_1) \cdots l_{12}^+(z_n) \Omega \rangle \\ &= \frac{1}{\hbar^n} \pi(z_1) \cdots \pi(z_n) \prod_{0 \leq i < j \leq n} \frac{z_j - z_i}{z_j - z_i - \hbar} \langle \tilde{\xi}, e(z_0) l_{12}^+(z_1) \cdots l_{12}^+(z_n) \Omega \rangle. \end{aligned}$$

Now

$$\begin{aligned} & \langle \tilde{\xi}, e(z_0) l_{12}^+(z_1) \cdots l_{12}^+(z_n) \Omega \rangle \\ &= -\frac{1}{\hbar} \langle \tilde{\xi}, (k^+(z_0) l_{12}^+(z_0) + k^-(z_0) l_{12}^-(z_0)) l_{12}^+(z_1) \cdots l_{12}^+(z_n) \Omega \rangle \\ &= -\frac{1}{\hbar} [\pi(z_0) \langle \tilde{\xi}, l_{12}^+(z_0) \cdots l_{12}^+(z_n) \Omega \rangle + \langle \tilde{\xi}, k^-(z_0) l_{12}^+(z_1) \cdots l_{12}^+(z_n) l_{12}^-(z_0) \Omega \rangle]. \end{aligned}$$

because  $l_{12}^-(z_0)$  commutes with the  $l_{12}^+(z_i)$ . Since  $e^-(z_0) \Omega = 0$ , we have  $l_{12}^-(z_0) \Omega = 0$ , which proves (66) at step  $n + 1$ . This shows (66).

Let us now show how (67) can be derived from (66). Let us set  $\tilde{\xi}(v) = \langle \xi, \psi(v) \rangle$ . Then we have (65) with  $\pi(z) = \pi_V(z)$ . Then

$$\begin{aligned} & \langle \psi(e(z_1) \cdots e(z_n) \Omega), \xi \rangle \\ &= \frac{1}{(-\hbar)^n} \prod_{i < j} \frac{z_j - z_i}{z_j - z_i - \hbar} \pi_V(z_1) \cdots \pi_V(z_n) \langle \psi(l_{12}^+(z_1) \cdots l_{12}^+(z_n) \Omega), \xi \rangle \\ &= \frac{1}{(-\hbar)^n} \prod_{i < j} \frac{z_j - z_i}{z_j - z_i - \hbar} \pi_V(z_1) \cdots \pi_V(z_n) \langle l_{12}^+(z_1) \cdots l_{12}^+(z_n) \psi(\Omega), \xi \rangle \\ &= \frac{1}{\hbar^n} \prod_{i < j} \frac{z_j - z_i}{z_j - z_i - \hbar} \pi_V(z_1) \cdots \pi_V(z_n) \langle \psi(\Omega), l_{12}^+(z_1 - \hbar) \cdots l_{12}^+(z_n - \hbar) \xi \rangle. \end{aligned}$$

(the first equality by (66); the second equality follows from the fact that  $\psi$  is a  $Y^{\geq 0}$ -map; the third equality follows by definition of action on  $V^*$  and because  $S(l_{12}^+(z)) = -l_{12}^+(z - \hbar)$ .  $\square$

**Corollary 11.1.** *We have*

$$\langle \psi[\tilde{e}(z_1) \cdots \tilde{e}(z_n)\Omega], \xi \rangle = \frac{1}{\hbar^n} \langle \psi(\Omega), l_{12}^+(z_1) \cdots l_{12}^+(z_n)\xi \rangle.$$

*Proof.* We have

$$\begin{aligned} & \langle \psi[\tilde{e}(z_1) \cdots \tilde{e}(z_n)\Omega], \xi \rangle \\ &= \langle \psi[k^+(z_1 + \hbar)^{-1}e(z_1 + \hbar) \cdots k^+(z_n + \hbar)^{-1}e(z_n + \hbar)\Omega], \xi \rangle \\ &= \prod_{i < j} (e(z_i), k^+(z_j)^{-1}) \prod_i \pi_V(z_i + \hbar)^{-1} \langle \psi[e(z_1 + \hbar) \cdots e(z_n + \hbar)\Omega], \xi \rangle \\ &= \frac{1}{\hbar^n} \prod_{i < j} (e(z_i), k^+(z_j)^{-1}) \prod_{i < j} \frac{z_j - z_i}{z_j - z_i - \hbar} \langle \psi(\Omega), l_{12}^+(z_1) \cdots l_{12}^+(z_n)\xi \rangle \\ &= \frac{1}{\hbar^n} \langle \psi(\Omega), l_{12}^+(z_1) \cdots l_{12}^+(z_n)\xi \rangle, \end{aligned}$$

where the first equality follows from (64), the second from the commutation rules, the next from Lemma 11.1.  $\square$

On the other hand, we have

$$\begin{aligned} & \langle \psi(T(z)\tilde{e}(z_1) \cdots \tilde{e}(z_n)v), \xi \rangle \\ &= \langle \psi(\tilde{e}(z_1) \cdots \tilde{e}(z_n)T(z)v), \xi \rangle \quad (\text{by centrality of } T(z)) \\ &= \frac{1}{\hbar^n} \langle \psi(T(z)\Omega), l_{12}^+(z_1 + \hbar) \cdots l_{12}^+(z_n + \hbar)\xi \rangle \end{aligned} \tag{68}$$

(by Lemma 11.1 above and because  $T(z)\Omega$  is killed by  $e^-(z)$ ).

Set  $L^\pm(z) = (l_{ij}^\pm(z))_{1 \leq i, j \leq 2}$ , then we have  $T(z) = \text{tr } L^+(z)L^-(z - 2\hbar)$  (see [14]). But since  $l_{ij}^-(z)v = \delta_{ij}v$ , we get  $T(z)v = (l_{11}^+(z) + l_{22}^+(z))v$ ; therefore the right side of (68) is equal to

$$\begin{aligned} & \frac{1}{\hbar^n} \langle \psi[(l_{11}^+(z) + l_{22}^+(z))\Omega], l_{12}^+(z_1) \cdots l_{12}^+(z_n)\xi \rangle \\ &= \frac{1}{\hbar^n} \langle \psi(\Omega), (l_{11}^+(z) + l_{22}^+(z))l_{12}^+(z_1) \cdots l_{12}^+(z_n)\xi \rangle \\ &= \frac{1}{\hbar^n} \hat{T}_z \{ \langle \psi(\Omega), l_{12}^+(z_1) \cdots l_{12}^+(z_n)\xi \rangle \} \end{aligned}$$

by Prop. 11.1. On the other hand,  $\langle \psi[T(z)\tilde{e}(z_1) \cdots \tilde{e}(z_n)\Omega], \xi \rangle$  is equal to

$$T_z^{(\Pi)} \{ \langle \psi(\Omega), l_{12}^+(z_1) \cdots l_{12}^+(z_n)\xi \rangle \} = \frac{1}{\hbar^n} T_z^{(\Pi)} \{ \langle \psi(\Omega), l_{12}^+(z_1) \cdots l_{12}^+(z_n)\xi \rangle \}$$

by Thm. 10.1. Since any symmetric polynomial can be realized as a correlation function  $\langle \psi(\Omega), l_{12}^+(z_1) \cdots l_{12}^+(z_n)\xi \rangle$ , we have shown:

**Proposition 11.2.** *The operators  $T_z^{(\Pi)}$  and  $\hat{T}_z$  are equal.*

This fact can also be verified by direct computation.

*Remark 11. Elliptic case.* In the elliptic case, and when there is no  $z_i$ ,  $T_z^{(\Pi=1)}$  is independent on  $z$  and coincides with the  $q$ -Lamé operator:

$$\hbar\theta(\hbar)(T_z f)(\lambda) = \frac{\theta(2\lambda - \hbar)}{\theta(2\lambda)} f(\lambda - \frac{\hbar}{2}) + \frac{\theta(2\lambda + \hbar)}{\theta(2\lambda)} f(\lambda + \frac{\hbar}{2}).$$

It should be possible to obtain the  $q$ -Lamé operator for  $m > 1$  with other  $\Pi$ .

#### APPENDIX A. DELTA-FUNCTION IDENTITIES

**Lemma A.1.** *We have*

$$q_-(z, w)^{-1} - q_+(z, w)^{-1} = \sigma(z)\delta(q^{-\partial}z, w) \quad (69)$$

and

$$q_-(q^{-\partial}z, w) - q_+(q^{-\partial}z, w) = -\sigma(z)\delta(q^{-\partial}z, w), \quad (70)$$

with  $\sigma$  defined by (43).

$\sigma$  has also the expression

$$\sigma(q^\partial z) = \left[ e^{2\sum_i (U+e_i)(z) \otimes e^i(w)} e^{-\phi(-\hbar, \partial_z^i \gamma)} \psi(-\hbar, \partial_z^i \gamma) \right]_{w=z} \quad (71)$$

*Proof.* From (6) follows that

$$q_-(q^\partial z, w)^{-1} - q_+(q^\partial z, w)^{-1} = -e^{-2\sum_i (q^\partial U + e_i)(z) \otimes e^i(w)} e^{-\phi(\hbar, \partial_z^i \gamma)} \psi(\hbar, \partial_z^i \gamma) \delta(z, w),$$

so that (69) follows, with  $\sigma$  given by (43).

Recall that we have

$$q(z, w) = i(z, w) \frac{q^{-\partial}z - w}{z - q^{-\partial}w},$$

with  $i(z, w)$  in  $\mathbb{C}[[z, w]][z^{-1}, w^{-1}][[\hbar]]^\times$  such that  $i(z, w)i(w, z) = 1$  ([8], Prop. 3.1). We have seen that

$$q_-(z, w) = i_+(z, w) \frac{q^{-\partial}z - w}{z - w},$$

with  $i_+(z, w)$  in  $\mathbb{C}[[z, w]][z^{-1}, w^{-1}][[\hbar]]^\times$ . Moreover,  $i_+(z, w)$  satisfies

$$i_+(z, w)i_+(q^\partial z, w) = \frac{q^\partial z - w}{z - q^{-\partial}w} i(z, w). \quad (72)$$

On the other hand, we have

$$q_+(z, w) = i_+(z, w) \frac{w - q^{-\partial}z}{w - z},$$

so that

$$q_-(q^\partial z, w)^{-1} - q_+(q^\partial z, w)^{-1} = i_+(q^\partial z, z)(q^\partial z - z)\delta(z - w),$$

and

$$q_-(z, w) - q_+(z, w) = i_+(z, z)(q^{-\partial}z - z)\delta(z - w),$$

with  $\delta(z - w) = \sum_{i \in \mathbb{Z}} z^i w^{-i-1}$ . From  $i(z, z) = 1$  and (72) follows that the prefactors of  $\delta(z - w)$  in both equations are opposite to each other. (70) follows.

On the other hand, we have

$$q_+(z, w) = q^{2\sum_i (U_+ e_i)(z) \otimes e^i(w)} e^{-\phi(-\hbar, \partial_z^i \gamma)} (1 + G^{(21)}(z, w) \psi(-\hbar, \partial_z^i \gamma)),$$

so that

$$q_-(z, w) - q_+(z, w) = \rho(q^\partial z) \delta(z, w).$$

with  $\rho$  given by

$$\rho(q^\partial z) = \left[ -e^{2\sum_i (U_+ e_i)(z) \otimes e^i(w)} e^{-\phi(-\hbar, \partial_z^i \gamma)} \psi(-\hbar, \partial_z^i \gamma) \right]_{w=z};$$

since  $\rho(q^\partial z)$  is equal to  $-\sigma(q^\partial z)$ , we get expression (71) for  $\sigma$ .  $\square$

**Lemma A.2.** *We have*

$$q_+(z, w)^{-1} G(w, z) + q_-(z, w)^{-1} G(z, w) = \alpha(z) \delta(q^{-\partial} z, w), \quad (73)$$

$$q_+(z, w) G(w, q^{-\partial} z) + q_-(z, w) G(q^{-\partial} z, w) = \beta(q^\partial z) \delta(z, w), \quad (74)$$

with  $\alpha$  and  $\beta$  defined by (44) and (45).  $\beta$  has also the expression

$$\beta(q^\partial z) = \left[ \partial_{\hbar} [e^{-\phi(-\hbar, \partial_z^i \gamma)} \psi(-\hbar, \partial_z^i \gamma)] e^{2\sum_i (U_+ e_i)(z) \otimes e^i(w)} \right]_{w=z}. \quad (75)$$

*Proof.* Let us prove (73). Applying  $q^\partial \otimes 1$  to this equation, we write it as

$$q_+(q^\partial z, w)^{-1} G(w, q^\partial z) + q_-(q^\partial z, w)^{-1} G(q^\partial z, w) = \alpha(q^\partial z) \delta(z, w).$$

We have

$$q_+(q^\partial z, w)^{-1} = e^{-q^\partial U_+ e_i(z) \otimes e^i(w)} e^{\sum_i \frac{1-q^\partial}{\partial} e_i(z) \otimes e^i(w)}.$$

From sect. 2.1 follows that

$$\begin{aligned} e^{\sum_i \frac{1-q^\partial}{\partial} e_i(z) \otimes e^i(w)} &= e^{-\phi(-\hbar, (-\partial_z)^i \gamma)} (1 - G^{(21)} \psi(-\hbar, (-\partial_z)^i \gamma)) \\ &= e^{-\phi(\hbar, \partial_z^i \gamma)} (1 + G^{(21)} \psi(\hbar, \partial_z^i \gamma)). \end{aligned} \quad (76)$$

$q_+(q^\partial z, w)^{-1} G^{(21)}(q^\partial z, w)$  is equal to

$$\begin{aligned} & q^{-2q^\partial (T_+ + U_+) e_i(z) \otimes e^i(w)} G^{(21)}(q^\partial z, w) \\ &= e^{(\frac{1-q^\partial}{\partial} - 2q^\partial U_+) e_i(z) \otimes e^i(w)} G^{(21)}(q^\partial z, w) \\ &= -e^{-2\sum_i q^\partial U_+ e_i(z) \otimes e^i(w)} \partial_{\hbar} (e^{\sum_i \frac{1-q^\partial}{\partial} e_i(z) \otimes e^i(w)}) \\ &= -e^{-2\sum_i q^\partial U_+ e_i(z) \otimes e^i(w)} \partial_{\hbar} [e^{-\phi(\hbar, \partial_z^i \gamma)} (1 + G^{(21)} \psi(\hbar, \partial_z^i \gamma))] \end{aligned}$$

therefore

$$\begin{aligned} & q_+(q^\partial z, w)^{-1} G^{(21)}(q^\partial z, w) - q_-(q^{-\partial} z, w)^{-1} G(q^\partial z, w) \\ &= -e^{-2 \sum_i q^\partial U_+ e_i(z) \otimes e^i(w)} \partial_{\hbar} [e^{-\phi(\hbar, \partial_z^i \gamma)} \psi(\hbar, \partial_z^i \gamma)] \delta(z, w). \end{aligned}$$

Therefore

$$\alpha(q^\partial z) = \left[ -e^{-\sum_i q^\partial U_+ e_i(z) \otimes e^i(w)} \partial_{\hbar} \{ e^{-\phi(\hbar, \partial_z^i \gamma)} \psi(\hbar, \partial_z^i \gamma) \} \right]_{z=w}.$$

Let us now prove (74). From sect. 2.1 follows that

$$e^{\sum_i (\frac{1-q^{-\partial}}{\partial} e_i)(z) \otimes e^i(w)} = e^{-\phi(\hbar, (-\partial_z)^i \gamma)} (1 - G^{(21)} \psi(\hbar, (-\partial_z)^i \gamma)). \quad (77)$$

Differentiating (77) with respect to  $\hbar$ , we find

$$G^{(21)}(q^{-\partial} z, w) e^{\sum_i (\frac{1-q^{-\partial}}{\partial} e_i)(z) \otimes e^i(w)} = \partial_{\hbar} [e^{-\phi(\hbar, (-\partial_z)^i \gamma)} (1 - G^{(21)} \psi(\hbar, (-\partial_z)^i \gamma))].$$

Therefore, we have also

$$G(q^{-\partial} z, w) (e^{\sum_i (\frac{1-q^{-\partial}}{\partial} e_i)(z) \otimes e^i(w)})_{w < z} = \partial_{\hbar} [e^{-\phi(\hbar, (-\partial_z)^i \gamma)} (-1 - G \psi(\hbar, (-\partial_z)^i \gamma))],$$

so that

$$\begin{aligned} & G^{(21)}(q^{-\partial} z, w) e^{\sum_i (\frac{1-q^{-\partial}}{\partial} e_i)(z) \otimes e^i(w)} - [G^{(21)}(q^{-\partial} z, w) e^{\sum_i (\frac{1-q^{-\partial}}{\partial} e_i)(z) \otimes e^i(w)}]_{w < z} \\ &= -\partial_{\hbar} [e^{-\phi(\hbar, (-\partial_z)^i \gamma)} \psi(\hbar, (-\partial_z)^i \gamma)] \delta(z, w), \end{aligned}$$

and

$$\begin{aligned} & G^{(21)}(q^{-\partial} z, w) q_+(z, w) + G(q^{-\partial} z, w) q_-(z, w) \\ &= -\partial_{\hbar} [e^{-\phi(\hbar, (-\partial_z)^i \gamma)} \psi(\hbar, (-\partial_z)^i \gamma)] e^{2 \sum_i (U_+ e_i)(z) \otimes e^i(w)} \delta(z, w), \end{aligned}$$

that is (74) with  $\beta$  given by (75). Identity (71) allows then to write  $\beta$  in the form (45).  $\square$

**Lemma A.3.** *We have*

$$G_{-2\lambda}(q^\partial w, z) q_-(q^{2\partial} w, q^\partial z)^{-1} + G_{2\lambda}(z, q^\partial w) q_+(q^{2\partial} w, q^\partial z)^{-1} = A_\lambda(z) \delta(z, w),$$

and

$$G_{-2\lambda}(q^{-\partial} w, z) q_-(q^\partial w, q^\partial z) + G_{2\lambda}(z, q^{-\partial} w) q_+(q^\partial w, q^\partial z) = B_\lambda(z) \delta(z, w),$$

where  $A_\lambda(z)$  and  $B_\lambda(z)$  are defined by (46) and (47).

*Proof.* Using Lemmas A.2 and A.1, we find

$$A_\lambda(z) = \alpha(q^{2\lambda} z) + [G_{-2\lambda}(q^\partial w, z) - G(q^{2\partial} w, q^\partial z)]|_{w=z} \sigma(q^{2\partial} z).$$

Then  $G(q^\partial w, z) - G(q^{2\partial} w, q^\partial z)$  is equal to

$$\sum_i e_i \otimes q^\partial e^i - \sum_i q^\partial e_i \otimes q^{2\partial} e^i; \quad (78)$$

the pairing of  $R$  with the first component of this tensor gives zero, so that it belongs to  $(R \otimes R)[[\hbar]]$  and its pairing with  $\lambda$  in  $\Lambda$  gives  $-q^{2\partial}(q^{-\partial}\lambda)_R$ ; therefore (78) is equal to

$$-\sum_i e^i \otimes q^{2\partial}(q^{-\partial}e_i)_R.$$

so that  $A_\lambda(z)$  is given by (46).

In the same way, we find

$$B_\lambda(z) = \beta(q^{2\partial}z) - \sigma(q^{2\partial}z)[G_{-2\lambda}(q^{-\partial}w, z) - G(w, q^\partial z)]|_{w=z}.$$

Since

$$G(q^{-\partial}w, z) - G(w, q^\partial z)$$

is equal to  $-\sum_i e^i(z)((q^{-\partial}e_i)_R)(z)$ , it follows that  $B_\lambda(z)$  is given by (47).  $\square$

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